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Properties of the positive solution of a semilinear elliptic partial differential equation in $\mathbb{R}^n$

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Abstract

A positive solution of a semilinear elliptic partial differential equation over the whole of $\mathbb{R}^n$ is shown to be a regular decay function, i.e. $\overline{u} \in C^2$ and $\lim_{|x|\to+\infty} \overline{u}(x) = 0$, by means of the Sobolev embedding theorem and a bootstrap argument.

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1. Introduction

We are concerned here with the properties of the nontrivial positive solution to the problem

$$Lu = \lambda g(x) u + f(x, u), \quad x \in \mathbb{R}^n, n \geq 3 \tag{1.1}$$

where $Lu = -\Delta u + q(x) u$ denotes the Schrödinger operator. The functions q and g are smooth and bounded in $\mathbb{R}^n$. The function q is positive but the function g changes sign (indefinite weight function) which means mes$(\{x \in \mathbb{R}^n : g(x) \gtreqless 0\}) > 0$.
We are interested with the case of a Carathéodory nonlinearity f satisfying a sublinear growth of the form

$$f(x, u) \leq a_\infty(x) u + \psi(x)$$

where functions a_∞ and ψ satisfy some hypotheses to be mentioned below.

We have proved in [5], by a constructive technique that we will recall in the next section, the existence of one weak positive solution of problem (1.1). It is the purpose of this paper to discuss the regularity and the behavior of this solution.

Some studies have been done in the linear case, for example [2], with a null potential q and a weight function g which is negative in the exterior of a ball.

We also mention the results obtained in [3] and [8] for the case of bounded domains.

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Here we will extend these results to our case and weaken the condition imposed on the weight function g in [2] by using weighted Sobolev spaces.

Before starting let us recall some results which one can find in [5].

2. Notation and recalling results

In [5] we have supposed that

(2.1) $\exists c_1 > 0, \exists \beta \geq 1 : 0 \leq q(x) \leq c_1(1+|x|^2)^{-\beta}$.
(2.2) $\exists c_2 > 0, \exists \theta > 1 : |g(x)| \leq c_2(1+|x|^2)^{-\theta}$ and $\mathrm{mes}(\{x \in \mathbb{R}^n : g(x) = 0\}) = 0$.
(2.3) f is a Carathéodory function.
(2.4) $f(x,0) > 0$ throughout $\mathbb{R}^n$.
(2.5) $u \mapsto f(x,u)$ is an increasing function in $[0,+\infty[$.
(2.6) $u \mapsto \frac{f(x,u)}{u}$ is a decreasing function in $[0,+\infty[$.
(2.7) The limit $\lim_{u\to+\infty} \frac{f(x,u)}{u} = a_\infty(x)$ exists uniformly for $x \in \mathbb{R}^n$.
(2.8) $f(x,u) \leq a_\infty(x)u + \psi(x), \forall u > 0$.
(2.8.a) $\exists c > 0, \exists \beta' \geq 1 : 0 \leq q(x) - a_\infty(x) \leq c p_{\beta'}(x)$.
(2.8.b) $a_\infty \in L^n_{p_1^{-\frac{n}{2}}}(\mathbb{R}^n)$.
(2.8.c) $0 < \psi \in L^2_{p_1^{-1}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ where $p_t(x) = (1+|x|^2)^{-t}$.

In addition we will suppose here that

(2.9) $\exists \alpha, 0 < \alpha < 1$, such that $q, g \in C^{0,\alpha}_{\mathrm{Loc}}(\mathbb{R}^n)$ and $f \in C^{0,\alpha}_{\mathrm{Loc}}(\mathbb{R}^n \times \mathbb{R})$.

We denote by $\mathbb{V}$ the weighted Sobolev space defined as the completeness of $C_0^\infty(\mathbb{R}^n)$ by means of the norm

$$\|u\|^2_{\mathbb{V}} = \int_{\mathbb{R}^n} \left(|\nabla u|^2 + \frac{1}{(1+|x|^2)}|u|^2 \right) \mathrm{d}x \tag{2.10}$$

where $\mathbb{V}$ is a Hilbert space (see [7]).

We denote by $W^{p,q}$ the usual Sobolev space.

Let us denote as $\lambda_1^+(q-a_\infty)$ the first positive eigenvalue of the linear problem

$$-\Delta v + (q - a_\infty)u = \lambda g v \quad \text{in } \mathbb{R}^n$$

which is principal (see [2]).

Theorem 2.1 (*Theorem 4.1[5]*). *If hypotheses* (2.1) *through* (2.8.c) *are verified, Eq.* (1.1) *admits an unique positive solution for any* λ *satisfying* $0 \leq \lambda < \lambda_1^+(q-a_\infty)$.

In the proof we have constructed a decreasing sequence starting with a supersolution u^0 of problem (1.1) chosen as the unique solution of the problem

$$-\Delta u^0 + (q - a_\infty)u^0 - \lambda g u^0 = \psi(x) \qquad \text{in } \mathbb{R}^n.$$

If we define formally the operator

$$T = (-\Delta + (q + \lambda g^-))^{-1}(\lambda g^+ + F) : \mathbb{V} \longrightarrow L^2_{p_1^{-1}}(\mathbb{R}^n) \tag{2.11}$$

where g^+ (resp. g^-) denotes the positive (resp. negative) part of the function g and F is the Nemytskii operator associated with the nonlinearity f, then the sequence is defined by

$$w_0 = u^0, \qquad w_1 = Tw_0, \qquad w_2 = Tw_1, \ldots, w_m = Tw_{m-1}, \ldots \quad m \geq 1. \tag{2.12}$$

We have shown that this is a strongly convergent sequence in $\mathbb{V}$, to a positive solution of problem (1.1), and by a maximal argument (Lemma 4.2 [5]) we proved its uniqueness.