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# Continuous selection of the solution map for one-sided Lipschitz differential inclusions

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**Abstract**

In the paper we deal with differential inclusions with one-sided Lipschitz (OSL) continuous right-hand sides, and prove the existence of a continuous selection of the solution map which assigns to any point the set of solutions to the multivalued Cauchy problem. As an application we study the problem of the existence of viable trajectories in a prescribed closed subset of a Banach space via the generalized Ważewski retract method.


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## 1. Introduction

The paper is motivated by (and partially deals with) the viability problem which consists in establishing the existence of a solution to the differential problem

$$\begin{cases} \dot{x}(t) \in F(x(t)) & \text{for a.e. } t \in I, \\ x(0) = x_0, \end{cases} \tag{1}$$

with $x(t) \in K$ for a prescribed set $K$ and every $t \geq 0$, if $I = [0, \infty)$. Such solutions are said to be *viable* in $K$. By a solution to (1) we always mean an absolutely continuous function $x$ satisfying $\dot{x}(t) \in F(x(t))$ almost everywhere (a.e.) in $I$. The set of all solutions to problem (1) will be denoted by $S_F(x_0)$.

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The best known and the most powerful tool for studying the viability problem is the famous Ważewski retract method ([23], Theorem 2) which, stated originally for differential equations (the single-valued case), can be generalized to some classes of multivalued problems (see, e.g., [20,17,19,13] where the homotopy index was used, and [14,10,12] where selection or approximation techniques were applied).

In [10] the authors generalize some results of [14] to the case of infinite-dimensional Banach spaces, where the Conley homotopy index is not defined, in general.[1] They prove the following

**Proposition 1.1** (*[10], Proposition 1*). *Let $F : E \multimap E$ be a Lipschitz*[2] *map with convex compact values, let $K \subset E$ be a closed subset, and assume that the exit set $K^-(F) = K_e(F)$ of $K$ (see Preliminaries) is closed in $K$.*

*If $K^-(F)$ is not a strong deformation retract*[3] *of $K$, then there exists a viable solution to (1) in $K$.*

Comparing this proposition with other results in [10] one can notice that the authors were forced to limit themselves to the class of Lipschitz maps because of the lack of continuous selectionability of the solution map $x_0 \mapsto S_F(x_0)$ for a larger class of one-sided Lipschitz multivalued maps (see Definition 2.1).

The first and the main goal of the present paper is to prove that the solution map $S_F(\cdot)$ admits a continuous selection if $F$ is a one-sided Lipschitz (OSL) continuous map (Theorem 3.1). Note that the continuity of $S_F(\cdot)$ can be checked using the generalized Filippov theorem (see, e.g., [8,9]) but no selection result enables us to infer from this fact the existence of a continuous selection of $S_F(\cdot)$. For instance, the technique presented in [4] seems to be impossible to use for several reasons. The spaces of integrable or absolutely continuous functions are not appropriate because we cannot successfully estimate an $L^1$-norm of velocities for solutions of differential inclusions with OSL right-hand sides. Moreover, probably there is no hope of obtaining any operator corresponding to problem (1) which could be a (multivalued) contraction with convex or decomposable values.[4]

We propose a step-by-step procedure for constructing a sequence of continuous maps convergent to a selection of $S_F(\cdot)$ (Section 3).

The existence of a continuous selection of the solution map can have numerous applications. We concentrate on the viability problem (Section 4) and develop it in two directions. On the one hand, we prove general results for maps which induce selectionability of $S_F(\cdot)$, and apply them to OSL differential inclusions. On the other hand, we weaken assumptions on the exit sets making them more appropriate and natural in a multivalued case.

We leave as an open problem the interesting question of AR-structure of the solution set $S_F(x_0)$ of (1) for OSL-continuous differential inclusions.

[1] We refer the reader to [22,15,16] where the Conley index is constructed for some classes of single-valued problems in Banach and Hilbert spaces.

[2] This means Lipschitzeanity w.r.t. the Hausdorff metric $d_H$, i.e., $d_H(F(x), F(y)) \le L|x - y|$ for some $L \ge 0$ and every $x, y \in E$.

[3] We say that $A \subset X$ is a *strong deformation retract* of $X$ if there exists a continuous homotopy (deformation) $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$, $h(x, 1) \in A$ for every $x \in X$, and $h(x, t) = x$ for every $x \in A$ and $t \in [0, 1]$. It is seen that $h(\cdot, 1)$ is a retraction of $X$ onto $A$.

[4] If such an operator existed, we would try to apply suitable results of [21] or [3].