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Symmetric submersions of $\mathbb{R}^n \to \mathbb{R}^m$

A. Díaz-Cano[a], F. González-Gascón[b], D. Peralta-Salas[b,*]

[a] *Departamento de Álgebra, Facultad de CC. Matemáticas, UCM, Spain*
[b] *Departamento de Física Teórica II, Facultad de CC. Físicas, UCM, Spain*

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Abstract

Submersions f such that $f^{-1}(0)$ contains a given fiber $\mathcal{F}$, and that are invariant under a family of vector fields $\mathfrak{s}$ leaving $\mathcal{F}$ invariant, are constructed. Examples for which a submersion of this kind cannot exist are also given. In the absence of a geometric theory of submersions f invariant under $\mathfrak{s}$, most of our treatment is analytic.

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1. Introduction

There has recently been some interest in C^∞ submersions $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $f : \mathbb{R}^3 \to \mathbb{R}$; Costa et al., cf. [6], constructed $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$ submersions with fibers $f^{-1}(a)$ ($a \in \mathbb{R}^2$) diffeomorphic to $\mathbb{R}$ and S^1; Watanabe, cf. [13], constructed $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$ submersions possessing any finite number of knots contained in some fiber; similarly Miyoshi, cf. [11], using techniques different from Watanabe's, proved that any link in $\mathbb{R}^3$ can be realized exactly as a fibre of f, and extended Watanabe's results to submersions $f : M \to \mathbb{R}^2$, M being an open orientable 3-manifold. Recently Hector and Peralta-Salas (private communication) have generalized the Watanabe and Miyoshi results to codimension m submanifolds of $\mathbb{R}^n$.

Let $\mathcal{F}$ be the set formed by a finite union of codimension m topologically closed oriented differential submanifolds (compact or not), with trivial normal bundle. Let $\mathfrak{s}$ be a finite set whose elements are $\mathbb{R}^n$-vector fields leaving $\mathcal{F}$ invariant. The question which arises is that of whether or not there can be found submersions $f : \mathbb{R}^n \to \mathbb{R}^m$ such that:

(i) they are invariant under $\mathfrak{s}$, see Section 2, and

(ii) $f^{-1}(0) = \mathcal{F}$.

As far as we know this problem has not been previously studied in the literature, and therefore all the results in this paper are new. The techniques that we use to approach the problem are mainly analytical. It would be nice to get results of topological or geometrical nature connecting topological properties of the sets $\mathcal{F}$ and $\mathfrak{s}$, and the numbers

* Corresponding address: Universidad Complutense de Madrid, Departamento de Fisica Teorica II, 28040 Madrid, Spain. Tel.: +34 913945198; fax: +34 913944557.

E-mail address: dperalta@fis.ucm.es (D. Peralta-Salas).

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(n, m), in order to get positive and negative results concerning the existence of submersions f symmetric under $\mathfrak{s}$ (and such that $f^{-1}(0) = \mathcal{F}$).

Submersions often appear in the field of mechanics in relation with the problem of integrating an $\mathbb{R}^p$ vector field X associated with Newton's equations of motion (p = phase-space dimension). A submersion f arises concerning a set of m C^1 first integrals f_i ($i = 1, \dots, m$) of X of maximal rank, i.e., $\text{rank}(\nabla f_i)_{i=1,\dots,m} = m$.

These submersions, whose components f_i are first integrals of X, can be useful, cf. [1–3], for getting information on the orbit structure of X, particularly when X is completely integrable (that is, when $m = n - 1$). When, in addition to this, the first integrals f_i of X satisfy $\mathcal{L}_S(f_i) = 0$, $i = 1, \dots, m$, for every vector field $S \in \mathfrak{s}$, the vector fields of $\mathfrak{s}$ and X are tangent to the fibers of f and a sub-dynamics $X|_{f^{-1}(a)}$ is defined on each fiber $f^{-1}(a)$ ($a \in \mathbb{R}^m$). Here $\mathcal{L}_S$ stands for the Lie derivative operator along the streamlines of S. Further progress on the properties of the sub-dynamics $X|_{f^{-1}(a)}$ can be achieved when X and $\mathfrak{s}$ are related in some way, for instance, cf. [1–3], when the vector fields in $\mathfrak{s}$ are symmetries of X, that is when

$$\mathcal{L}_S(X) = \lambda(x) \cdot X.$$

In this case $X|_{f^{-1}(a)}$ is symmetric under the vector fields in $\mathfrak{s}$ restricted to $f^{-1}(a)$ and the reduction process $X \to X|_{f^{-1}(a)}$ can continue (via new submersions $\tilde{f} : f^{-1}(a) \to \mathbb{R}$) invariant under the vector fields in $\mathfrak{s}|_{f^{-1}(a)}$.

The organization of this paper is as follows. In Sections 2 and 3 we review the most relevant results on submersions theory and state the problem and main definitions of this work. For $n = 2, 3$ and $m = 1, 2$, conditions and examples for which the symmetries $\mathfrak{s}$ of $\mathcal{F}$ can or cannot be inherited by submersions f realizing $\mathcal{F}$ are given in Sections 4–7. Finally some open problems are discussed in Section 8. Most of the results obtained in this paper admit generalization to submersions $f : \mathbb{R}^n \to \mathbb{R}^m$, but this generalization is left to the reader.

2. Definition of the problem

Let $\mathcal{F}$ be the finite union of codimension m C^∞ orientable manifolds M embedded in $\mathbb{R}^n$. Throughout this paper these manifolds will be open or closed when $m > 1$ and open when $m = 1$. The following results on $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ submersions are quoted:

(i) For when $m = 1$ and $n = 3$ Hector and Bouma, cf. [9], have proved that any open surface S (properly) embedded in $\mathbb{R}^3$ can be realized as the zero level set of a submersion $f : \mathbb{R}^3 \to \mathbb{R}$.

(ii) For when $m = 2$ and $n = 3$ Miyoshi's techniques, cf. [11], allow one to construct submersions $f : \mathbb{R}^3 \to \mathbb{R}^2$ when $\mathcal{F}$ is a finite union of topological circles.

(iii) On the other hand, for when $m = n - 1$, $n > 3$ and $\mathcal{F}$ consists of a finite number of topological straight lines, it is not difficult to prove that $\mathcal{F}$ is ambient diffeomorphic to a set L_j of parallel straight lines lying on a two-dimensional plane $\Pi_2 \subset \mathbb{R}^n$ (say the plane $x_i = 0$ for $i = 3, \dots, n$). A submersion $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$ can then be easily constructed from the $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ submersion defined by

$$\left.\begin{array}{l} \tilde{f}(x_1, x_2) = a(x_1) b(x_2) \\ a(x_1) = 0 \Leftrightarrow x_1 = \lambda_j, \quad j = 1, \dots, N \\ a'(\lambda_j) \neq 0, \quad j = 1, \dots, N \\ b \neq 0, b' \neq 0 \end{array}\right\},$$

L_j being given by $x_1 = \lambda_j$. The reader will easily check that $\tilde{f}$ is a submersion and that $\tilde{f}^{-1}(0) = \mathcal{F} = \bigcup_{j=1}^N L_j$.

Let $\mathfrak{s}$ be a finite set formed by C^∞, $\mathbb{R}^n$-vector fields (v.f. in what follows) tangent to $\mathcal{F}$ (that is, the connected components of $\mathcal{F}$ are invariant under the v.f. in $\mathfrak{s}$). In most of the applications considered in this paper the set $\mathfrak{s}$ will be closed under the Lie–Jacobi bracket, but this property does not necessarily hold (see Section 7.ii.2).

We now formulate our problem: Get a submersion $f : \mathbb{R}^n \to \mathbb{R}^m$ invariant under $\mathfrak{s}$ having $\mathcal{F}$ as its zero level set ($f^{-1}(0) = \mathcal{F}$).

The invariance of f under $\mathfrak{s}$ can be analytically expressed in the form

$$\left.\begin{array}{l} \mathcal{L}_S(f_i) = F_i(f_1, \dots, f_m) \\ i = 1, \dots, m \\ \forall S \in \mathfrak{s} \end{array}\right\}, \tag{1}$$