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ScienceDirect
Nonlinear Analysis 65 (2006) 1891–1912
Nonlinear Analysis
www.elsevier.com/locate/na

Multiplicity and concentration of solutions for nonhomogeneous elliptic equations in multi-bump domains

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Received 8 April 2005; accepted 27 October 2005

Abstract

In this paper, we study the decomposition of the filtration of the Nehari manifold via the variation of domain shape. Furthermore, we use this result to prove that the nonhomogeneous elliptic equations in an m-bump domain have at least $m + 1$ positive solutions.

Keywords: Semilinear elliptic equations; Palais–Smale; Multi-bump domains

1. Introduction

In this paper, we study the multiplicity and concentration of positive solutions of the following equations:

$$\begin{cases} -\Delta u + u = |u|^{p-2}u + h(x) & \text{in } \Omega; \\ u \in H_0^1(\Omega), \end{cases} \tag{E_h}$$

where Ω is a domain in $\mathbb{R}^N, 2 < p < \frac{2N}{N-2} (N \geq 3), 2 < p < \infty (N = 2)$ and $h(x) \in H^{-1}(\Omega) \backslash \{0\}, h(x) \geq 0$. Here $H_0^1(\Omega)$ denotes the usual Sobolev space. Associated with Eq. (E_h), we consider the energy functional J_h in the Sobolev space $H_0^1(\Omega)$:

$$J_h(u) = \frac{1}{2} \int_{\Omega} \left(|\nabla u|^2 + u^2 \right) - \frac{1}{p} \int_{\Omega} |u|^p - \int_{\Omega} hu.$$

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doi:10.1016/j.na.2005.10.043

It is well known that the solutions of Eq. (E_h) and the critical points of the energy functional J_h in $H_0^1(\Omega)$ are the same (see [17, Proposition B.10.]).

That the existence of multiple positive solutions of Eq. (E_h) has been the focus of a great deal of research in recent years. There has been some progress for the multiplicity of positive solutions of Eq. (E_h) as follows: Zhu [22] for $\Omega = \mathbb{R}^N$, Hsu and Wang [9] for Ω an exterior strip domain and Wang [21] for Ω an Exteban–Lions domain with holes. They used the concentration compactness of Lions [16] to prove the Eq. (E_h) has at least two positive solutions for $\|h\|_{L^2}$ sufficiently small and $h(x)$ exponentially decaying.

Actually, Cao and Zhou [6], Jeanjean [10] and Adachi and Tanaka [1,2] got the same result for the following general problems:

$$\begin{cases} -\Delta u + u = b(x)|u|^{p-2}u + h(x) & \text{in } \mathbb{R}^N; \\ 0 < u \in H^1(\mathbb{R}^N), \end{cases} \tag{$E_{b,h}$}$$

where $b(x) \in C(\mathbb{R}^N)$ and $b(x) \to b^\infty > 0$ as $|x| \to \infty$; in [6] and [10] for $b(x) \geq b^\infty$ and $\|h\|_{H^{-1}}$ sufficiently small; in Adachi and Tanaka [2] for $b(x) \geq b^\infty - Ce^{-\lambda|x|}$ for some $C, \lambda > 0$ and $\|h\|_{H^{-1}}$ sufficiently small; moreover, in Adachi and Tanaka [1] for $b(x) \leq b^\infty$ with strict inequality on a set of positive measure, showing at Eq. $(E_{b,h})$ has at least four positive solutions for $\|h\|_{H^{-1}}$ sufficiently small.

Note that our Eq. (E_h) can be regarded as a perturbation problem for the following homogeneous equation:

$$\begin{cases} -\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{E_0}$$

There has been some progress for the existence of solutions of Eq. (E_0) in unbounded domains as follows: Berestycki and Lions [5] for $\Omega = \mathbb{R}^N$, Chen and Wang [7] for Ω an interior flask domain, Lien, Tzeng and Wang [15] for Ω a periodic domain, Del Pino and Felmer [11,12] for Ω a quasicylindrical domain and Wu [21] for Ω a multi-bump domain. They proved that Eq. (E_0) in Ω has a ground state solution. It is well known that the ground state solutions of Eq. (E_0) in Ω can be obtained via the Nehari minimization problem

$$\alpha_0(\Omega) = \inf_{v \in \mathbf{M}_0(\Omega)} J_0(v), \tag{1}$$

where $\mathbf{M}_0(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} \mid \langle J_0'(u), u\rangle = 0\}$. Note that $\alpha_0(\Omega) > 0$ and if $u_0 \in \mathbf{M}_0(\Omega)$ achieves $\alpha_0(\Omega)$, then u_0 is a ground state solution of Eq. (E_0) in Ω (see [20] or [19]).

Moreover, when $\Omega = \mathbb{R}^N\backslash\omega$ is an exterior domain, where ω is a bounded domain. It is well known that Eq. (E_0) in $\mathbb{R}^N\backslash\omega$ does not admit any ground state solution (see [4]). However, Bahri and Lions [3] and Benci and Cerami [4] asserted that Eq. (E_0) in $\mathbb{R}^N\backslash\omega$ has a positive higher energy solution. When Ω is an Esteban–Lions domain, Eq. (E_0) in Ω does not admit any nontrivial solution (see [14]).

The main purpose of this paper is using the results of Wu [21] to prove that Eq. (E_h) in an m-bump domain has at least $m + 1$ positive solutions. Before stating our main result, we review some known results from [21]. Take $k \geq 1$ and assume the domains $\Theta_1, \Theta_2, \ldots, \Theta_k$ satisfy the following conditions.

(Θ1) $\Theta_i \cap \Theta_j = A_{i,j}$ is bounded for all $i \neq j$.

(Θ2) There is an $m \in \{1, 2, \ldots, k\}$ such that

$$\alpha_0(\cup_{i=1}^k \Theta_i) = \alpha_0(\Theta_1) = \cdots = \alpha_0(\Theta_m) < \alpha_0(\Theta_j) \qquad \text{for all } j = m+1, \ldots, k.$$