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Existence and uniqueness results for impulsive functional differential equations with scalar multiple delay and infinite delay

Abdelghani Ouahab

Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie

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Abstract

In this paper, we discuss local and global existence and uniqueness results for first order impulsive functional differential equations with multiple delay. We shall rely on a nonlinear alternative of Leray–Schauder. For the global existence and uniqueness we apply a recent nonlinear alternative of Leray–Schauder type in Fréchet spaces, due to M. Frigon and A. Granas [Résultats de type Leray–Schauder pour des contractions sur des espaces de Fréchet, Ann. Sci. Math. Québec 22 (2) (1998) 161–168]. The goal of this paper is to extend the problems considered by A. Ouahab [Local and global existence and uniqueness results for impulsive differential equations with multiple delay, J. Math. Anal. Appl. 323 (2006) 456–472].

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1. Introduction

This paper is concerned with the existence and uniqueness of solutions, for first order functional differential equations with impulsive effects and multiple delay. In Section 3, we will consider local existence and uniqueness results for first order impulsive functional differential equations with fixed moments and multiple delay

$$y'(t) = f(t, y_t) + \sum_{i=1}^{n_*} y(t - T_i), \quad \text{a.e. } t \in J := [0, b] \setminus \{t_1, t_2, \dots, t_m\}, \tag{1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{3}$$

where $n_* \in \{1, 2, \dots\}$, $f : J \times B \to \mathbb{R}^n$ is a given function, ($\phi \in B$ is called a *phase space* that will be defined later) $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, \dots, m$, are given functions satisfying some assumptions that will be specified later. For every $t \in [0, b]$, the history function $y_t : (-\infty, 0] \to \mathbb{R}^n$ is defined by

$$y_t(\theta) = y(t + \theta), \quad \text{for } \theta \in (-\infty, 0].$$

We assume that the histories $y_t : (-\infty, 0] \to \mathbb{R}^n$, $y_t(\theta) = y(t + \theta)$, belong to the abstract phase space B.

E-mail address: ouahab@univ-sba.dz.

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In the literature devoted to equations with finite delay, the state space is much of the time the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology; see the book of Hale and Lunel [22]. When the delay is infinite, the selection of the state B (i.e. phase space) plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [21] (see also Kappel and Schappacher [24] and Schumacher [37]) and the papers of Hale [19] and Sawano [36]. For a detailed discussion on this topic we refer the reader to the book by Hal [20] and Hino et al. [23]. For the case where the impulses are absent, an extensive theory for first order functional differential equations has been developed. We refer to Hale and Kato [21], Hale and Lunel [22], Corduneanu and Lakshmikantham [10], Hino et al. [23], Lakshmikantham et al. [26] and Shin [38].

Recently, the theory and application of impulsive delay differential equations have developed. Various mathematical models in the study of population dynamics, biology, ecology and epidemic, etc. can be expressed by impulsive delay differential equations. These processes and phenomena, for which the adequate mathematical models are impulsive delay differential equations, are characterized by the fact that there is sudden changing of their state and that the processes under consideration depend on their prehistory at each moment of time. An application of impulsive functional differential equations with multiple delay arises in the study of pulse vaccination strategies. In [14] the authors consider the following model:

$$\begin{cases} S'(t) = b - bS(t) - \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} + \gamma I(t-\tau)e^{-b\tau} \\ E'(t) = \displaystyle\int_{t-\omega}^{t} \frac{\beta S(u)I(u)}{N(u)} e^{-b(t-u)}\mathrm{d}u \\ I'(t) = \dfrac{\beta e^{-b\omega} S(t-\omega)I(t-\omega)}{1+\alpha S(t-\omega)} - (b+\omega)I(t), \\ R'(t) = \displaystyle\int_{t-\omega}^{t} \gamma I(u)e^{-b(t-u)}\mathrm{d}u \\ S(t_k^+) = (1-\theta)S(t_k^-), \quad t = kT, k \in \mathbb{N}, \\ E(t_k^+) = E(t_k^-), \quad t = kT, k \in \mathbb{N}, \\ I(t_k^+) = I(t_k^-), \quad t = kT, k \in \mathbb{N}, \\ R(t_k^+) = R(t_k^-) + \theta S(t_k^-), \quad t = kT, k \in \mathbb{N}, \end{cases} \tag{4}$$

where $\mathbb{N} = \{0, 1, 2, \dots, \}$, $N(t) = S(t) + N(t) + I(t) = 1$, for all $t \geq 0$ and

- (S) denotes the susceptible,
- (I) the infectives,
- (R) the removed group,
- (E) the exposed but not yet infectious.

Since time delay applies in many fields in our society, systems with time delay have received significant attention in recent years, for instance see the papers of Li and Huo [29], Tang and Chen [40], Yan et al. [45], and Zhang and Fan [46].

For the theory of impulsive differential equations, and impulsive delay differential equations we refer to the monographs of Benchohra et al. [5], Bainov and Simeonov [2], Lakshmikantham et al. [25] and Samoilenko and Perestyuk [35] and the papers of Agur et al. [1], Ballinger and Liu [3], Benchohra et al. [4], Chen and Su [8], Franco et al. [12], Li et al. [27], Li and Hu [29], Rogovchenko [33], Qian and Li [34], Li et al. [28], Xu et al. [42], Zhang et al. [47] and the references therein.

Systems with infinite delay deserve study because they describe a kind of system present in the real word. For example, in a predator–prey system the predation decreases the average growth rate of the prey species, to mature for particular duration of time (which for simplicity in mathematical analysis has been assumed to be infinite) before they are capable of decreasing the average growth rate of the prey species.

And there are some results on impulsive functional differential equations with infinite delay; see Benchohra et al. [6,7]. The goal of this paper is to give existence and uniqueness results for first order impulsive functional differential equations with infinite delay. The main theorems of this paper extend to the infinite delay problems considered by Benchohra et al. [4], Ouahab [32].