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ScienceDirect
Nonlinear Analysis 67 (2007) 1042–1054
Nonlinear Analysis
www.elsevier.com/locate/na

# On the existence of periodic solutions to Rayleigh differential equation of neutral type in the critical case[☆]

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Received 22 October 2005; accepted 26 June 2006

**Abstract**

In this paper, the authors study the existence of periodic solutions for a second order neutral functional differential equation

$$(x(t) - cx(t-\tau))'' = f(x'(t)) + g(t, x(t-\mu(t))) + e(t)$$

in the critical case $|c| = 1$. By analyzing some properties of the linear difference operator $A : [Ax](t) = x(t) - cx(t-\tau)$ and using Mawhin's continuation theorem, some new results are obtained.


*MSC:* 34B15; 34K13

*Keywords:* Periodic solution; Continuation theorem; Deviating argument

## 1. Introduction

In recent years, we noticed that the existence of periodic solutions for the following types of neutral differential equation with deviating arguments

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - bx(t-\tau)) = -ax(t - r + \gamma h(t, x(t+\cdot))) + f(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t) - ku(t-\tau)) = g_1(u(t)) + g_2(u(t-\tau_1)) + p(t)$$

and

$$(u(t) - ku(t-\tau))'' + f(u(t))u'(t) + \sum_{j=1}^{n} \beta_j(t) g(u(t-\gamma_j(t))) = p(t),$$

☆ Sponsored by the NNSF of China (10371006); the NSF of Anhui Province of China (2005kj031ZD; 050460103) and the Teaching and Research Award Program for Excellent Teachers in Higher Education Institution of Anhui Province of China.

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0362-546X/$ - see front matter 
doi:10.1016/j.na.2006.06.032

were studied by papers [1–3]. But the conditions of constants $|b| \neq 1$ in [1] and $|k| \neq 1$ in [2,3] were required. For example, in [2,3], under the assumption $|k| \neq 1$, we obtained that $A : C_{2\pi} := \{x : x \in C(R, R), x(t + 2\pi) \equiv x(t)\} \to C_{2\pi}$, $[Ax](t) = x(t) - kx(t - \tau)$ has a unique inverse $A^{-1} : C_{2\pi} \to C_{2\pi}$ defined by

$$[A^{-1} f](t) = \begin{cases} \sum_{j\geq 0} k^j f(t - j\tau), & |k| < 1, \\ -\sum_{j\geq 1} k^{-j} f(t + j\tau), & |k| > 1, \end{cases}$$

and then

$$\int_0^{2\pi} |[A^{-1} f](t)| \mathrm{d}t \leq \frac{1}{|1 - |k||} \int_0^{2\pi} |f(t)| \mathrm{d}t, \quad \forall f \in C_{2\pi}, \tag{1.1}$$

which was crucial for estimating *a priori bounds* of periodic solutions in the no-critical case $|k| \neq 1$. Under the critical condition $|c| = 1$, we studied a first order neutral differential equation

$$(x(t) - cx(t - \tau))' = g(t, x(t - \mu(t))) + e(t),$$

a Duffing differential equation of neutral type

$$(x(t) - cx(t - \tau))'' = g(t, x(t - \mu(t))) + e(t),$$

and a Liénard differential equation of neutral type

$$(x(t) - cx(t - \tau))'' = f(x(t))x'(t) + g(t, x(t - \mu(t))) + e(t)$$

in [4–6], respectively.

In this paper, we continue to study a Rayleigh differential equation of neutral type

$$(x(t) - cx(t - \tau))'' = f(x'(t)) + g(t, x(t - \mu(t))) + e(t), \tag{1.2}$$

where $f \in C(R, R)$, $g \in C(R \times R, R)$ with $g(t + 2\pi, x) \equiv g(t, x), \forall x \in R$, $e$ and $\mu$ are continuous periodic functions with period $2\pi$, $c, \tau \in R$ are two constants with $|c| = 1$. By employing Mawhin's continuation theorem and topological degree theory, we obtain some new results on the existence of $2\pi$-periodic solutions to Eq. (1.2).

The significance is that the approaches used for estimating *a priori bounds* of periodic solutions in the present paper are different from the corresponding ones in [1–6]. This is mainly due to the fact that here, on the one hand formula (1.1) which is crucial for obtaining *a priori bounds* of periodic solution in [1–3] does not hold in the critical case $|c| = 1$, and on the other hand, the crucial step $\int_0^{2\pi} f(x(s))x'(s)\mathrm{d}s = 0$ which is required in [5] for estimating *a priori bounds* of periodic solution is no longer valid for Eq. (1.2).

Throughout this paper, $Z$ is the set of integers, $Z_1$ is the set of odd integers, $Z_2$ is the set of even integers, $N$ is the set of positive integers, $N_1$ is the set of odd positive integers and $N_2$ is the set of even positive integers. Let $C_{2\pi} = \{x : x \in C(R, R), x(t + 2\pi) \equiv x(t)\}$ with the norm $|\varphi|_0 = \max_{t\in[0,2\pi]} |\varphi(t)|$, $C_{2\pi}^{+} = \{x : x \in C_{2\pi}, x(t + \pi) \equiv x(t)\}$, $C_{2\pi}^{0} = \{x : x \in C_{2\pi}, \int_0^{2\pi} x(s)\mathrm{d}s = 0\}$, $C_{2\pi}^{+,0} = \{x : x \in C_{2\pi}^{+}, \int_0^{2\pi} x(s)\mathrm{d}s = 0\}$, and $C_{2\pi}^{-} = \{x : x \in C_{2\pi}, x(t + \pi) \equiv -x(t)\}$ equipped with the norm $|\cdot|_0$; $C_{2\pi}^{1} = \{x : x \in C^1(R, R), x(t + 2\pi) \equiv x(t)\}$ with the norm $|\varphi|_{C_{2\pi}^1} = \max\{|\varphi|_0, |\varphi'|_0\}$; $L_{2\pi}^2 = \{x : x(t) \stackrel{\text{a.e.}}{=} x(t + 2\pi), t \in R$ and $\int_0^{2\pi} |x(s)|^2\mathrm{d}s < +\infty\}$, $L_{2\pi}^{2+} = \{x : x \in L_{2\pi}^2, x(t) \stackrel{\text{a.e.}}{=} x(t + \pi), t \in R\}$, $L_{2\pi}^{2-} = \{x : x \in L_{2\pi}^2, x(t) \stackrel{\text{a.e.}}{=} -x(t + \pi), t \in R\}$, and the norm is defined by $|\varphi|_2 = \left(\int_0^{2\pi} |\varphi(s)|^2\mathrm{d}s\right)^{1/2}$. Clearly, $C_{2\pi}^1$, $C_{2\pi}^-$, $C_{2\pi}$, $C_{2\pi}^+$, $C_{2\pi}^0$, $C_{2\pi}^{+,0}$, $L_{2\pi}^2$, $L_{2\pi}^{2+}$ and $L_{2\pi}^{2-}$ are all Banach spaces. Further, we denote $\bar{h} = \frac{1}{2\pi} \int_0^{2\pi} h(s)\mathrm{d}s$, $|h|_1 = \int_0^{2\pi} |h(s)|\mathrm{d}s, \forall h \in L_{2\pi}^2$.

## 2. Main lemmas

In this section, we analyze some properties of the linear difference operator $A : [Ax](t) = x(t) - cx(t - \tau)$, which will be used to estimate *a priori bounds* of periodic solutions in Section 3.