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The boundary value problem for quasilinear degenerate and singular elliptic systems

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Abstract

In this paper, using the contraction mapping principle and the shooting method, we obtain the existence and uniqueness of the local solutions and the global solutions to a class of quasilinear degenerate elliptic systems with the p-Laplacian-like as its principal and with singularity on the boundary. We also obtain the continuous dependence of the solution on the boundary data. Two examples are given to illustrate the applications of the theorems.

MSC: 35J55; 35J60

Keywords: Elliptic system; Existence and uniqueness; Contraction mapping principle; Shooting method

1. Introduction

During the past decade research on the positive radial solutions of the nonlinear equations with the *Laplacian* and the *p-Laplacian* as their principal parts has been very active (cf. [2–7] and references therein). Recently, it has begun to discuss the same kind of problems for semilinear equations (cf. [1]). But we notice that there are few articles on the same kind of problems for quasi-linear elliptic systems. Recently, an article [1] discussed the existence and uniqueness of

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positive solutions for a semilinear elliptic system

$$\begin{cases} -\Delta u = g(x, v), & x \in \Omega \\ -\Delta v = f(x, u), & x \in \Omega \\ u = v = 0, & x \in \partial\Omega \end{cases}$$

where the principal part is Laplacian with no degeneracy and no singularity therein. Also, the nonlinear terms are only related to the unknown functions.

Motivated by the same kind of problems for the single equation we discuss the existence and uniqueness of the local positive radial solution and the global positive radial solution for the following elliptic system

$$\begin{cases} \operatorname{div}\{A(|Du|)Du\} + f(x, u, v, |Du|, |Dv|) = 0, & x \in B \\ \operatorname{div}\{A(|Dv|)Dv\} + g(x, u, v, |Du|, |Dv|) = 0, & x \in B \\ u > 0, \quad v > 0, & x \in B \\ u = 0, \quad v = 0, & x \in \partial B \end{cases} \tag{1.1}$$

where $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$, $A : [0, \infty) \to \mathbb{R}$ is a continuous function and f, g are given functions depending on not only the unknown functions u and v but also the gradients of u and v. Also, f and g may have singularity on the boundary of B. One can notice that system (1.1) is degenerate at the points x where $Du(x) = 0$. Although the motivation is from the single equation it is more difficult to solve a system than a single equation since the nonlinear couple terms f and g appeared in the system. Also, we use different methods and theories from [1] to prove our main results in this paper.

We are interested in the positive radial solutions of (1.1). For this case, (1.1) reads

$$\begin{cases} (r^{n-1}H(u'(r)))' + r^{n-1}F(r, u(r), v(r), |u'(r)|, |v'(r)|) = 0, & 0 < r < 1 \\ (r^{n-1}H(v'(r)))' + r^{n-1}G(r, u(r), v(r), |u'(r)|, |v'(r)|) = 0, & 0 < r < 1 \\ u(r) > 0, \quad v(r) > 0, & 0 \le r < 1 \\ u'(0) = v'(0) = 0, \quad u(1) = v(1) = 0 \end{cases} \tag{1.2}$$

where $r = |x|$ and $H(t) = tA(|t|)$ for $t \in \mathbb{R}$. We shall prove the existence and uniqueness of the positive global solution of (1.2) in Section 3. To this end we first consider, in Section 2, the initial value problem

$$\begin{cases} (r^{n-1}H(u'(r)))' + r^{n-1}F(r, u(r), v(r), |u'(r)|, |v'(r)|) = 0, & 0 < r < 1 \\ (r^{n-1}H(v'(r)))' + r^{n-1}G(r, u(r), v(r), |u'(r)|, |v'(r)|) = 0, & 0 < r < 1 \\ u(0) = \alpha, \quad v(0) = \beta \\ u'(0) = v'(0) = 0 \end{cases} \tag{1.3}$$

where α, β are any positive numbers. Throughout this paper we suppose that

(A1) $F, G : [0, 1) \times [0, 1) \times (0, \infty) \times (0, \infty) \times [0, \infty) \to (0, \infty)$ are continuous and $F(x, y, z, y', z')$, $G(x, y, z, y', z')$ are locally Lipschitz continuous with respect to y, z, y' and z'.

(A2) $F(x, y, z, y', z')$ and $G(x, y, z, y', z')$ are strictly decreasing in y, z.

(A3) $F(x, y, z, y', z')$, $G(x, y, z, y', z')$ are strictly increasing in y', z'.

(A4) $A(t)$ is a continuous function on $(0, \infty)$.