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Periodic solutions to impulsive differential inclusions with constraints☆

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Abstract

The existence of a periodic solution to an impulsive differential inclusion being invariant with respect to a non-convex set of state constraints is established by the use a Lefschetz type fixed-point theorem for set-valued maps.

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1. Introduction

We study the existence of periodic solutions to the following impulsive state constraints problem

$$\begin{cases} u'(t) \in F(t, u(t)) & \text{a.e. } t \in [0, T] \setminus \{t_1, \ldots, t_n\}, \\ u(t_k^+) \in \psi_k(u(t_k)) & \text{for any } k \in \{1, \ldots, n\}, \\ u(t) \in K & \text{for } t \in [0, T], \\ u(0) = u(T), \end{cases} \tag{1}$$

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where $K \subset \mathbb{R}^d$ is a strictly regular compact set,[1] $F : [0, T] \times \mathbb{R}^d \multimap \mathbb{R}^d$ is a set valued map, $0 < t_1 < \cdots < t_n < T$ are the impulse times and $\psi_i : K \multimap K$ are the impulse maps.[2] The method we use involves results of two types: Lefschetz type fixed-point theorems for set-valued maps (see Theorem 2.13) and a topological characterization of the set of viable solutions to a differential inclusion (see Theorem 3.3). A similar approach has been used to obtain the existence of a periodic viable solution to a constrained differential inclusion (compare [4,19]). We extend this method to systems with impulses. To this aim, we provide a construction of the Lefschetz number for a class of set-valued maps with the so-called *decomposable factorization*. The construction uses a graph approximation theorem (Theorem 2.5).

The problem (1) was studied in e.g. [6,13,22], where the constraining set K was assumed to be compact and convex (see also [3] for some general constructions). The topological tools presented in the second section of the paper allow one to get a much more general result by the use of less technical arguments.

2. Topological setting

By a space we always mean a *metric* space; all single-valued maps between spaces are considered to be continuous. Given a space X with a metric d, a set $A \subset X$ and $\varepsilon > 0$, $B(A, \varepsilon) := \{x \in X \mid d_A(x) := \inf_{a \in A} d(x, a) < \varepsilon\}$ denotes the (open) *ε-neighborhood* of A. Recall that a space X is an ANR (*absolute neighborhood retract*) (we also write $X \in$ ANR) if, given a space Y and a homeomorphic embedding $i : X \to Y$ of X onto a closed subset $i(X) \subset Y$, $i(X)$ is a *neighborhood retract* of Y, i.e. there is an open neighborhood U of $i(X)$ in Y and a retraction $r : U \to i(X)$.[3]

It is well-known that the class of ANR behaves well concerning the fixed-point theory. The following result will be of crucial importance for our reason.

Theorem 2.1 (*Comp.* [21,5]). *If X is a compact ANR, $f : X \to X$ is a map, and the Lefschetz number $\lambda(f) \neq 0$, then f has a fixed point.* □

In order to understand this result, recall that any compact ANR is homotopy dominated by (or even, in view of the celebrated theorem of West, homotopy equivalent to) a compact (hence finite) polyhedron (see [5, Th. III.B.1]). Therefore, if $H_*(\cdot; \mathbb{Q})$ denotes *any* (for example, singular) ordinary homology functor with rational coefficients, then the (graded) vector space $H_*(X; \mathbb{Q})$ is of finite type, i.e. for each $q \geq 0$, the *Betti number* $\beta_q(X) := \dim_{\mathbb{Q}} H_q(X; \mathbb{Q}) < \infty$ and, for almost all $q \geq 0$, $\beta_q(X) = 0$ (comp. [5, Cor. III.B.2]). In particular, for each $q \geq 0$, the trace $\operatorname{tr} f_{*q}$ of the endomorphism $f_{*q} : H_q(X; \mathbb{Q}) \to H_q(X; \mathbb{Q})$ induced by f is a well-defined rational number and, for almost all $q \geq 0$, $\operatorname{tr} f_{*q} = 0$. Hence the *Lefschetz number*

$$\lambda(f) := \sum_{q \geq 0} (-1)^q \operatorname{tr} f_{*q}$$

is well-defined. Similar arguments show that the *Euler characteristic*

$$\chi(X) := \sum_{q \geq 0} (-1)^q \beta_q(X),$$

[1] The class of strictly regular sets will be described below; in particular, any compact convex set, as well as compact epi-Lipschitz set, is strictly regular.

[2] Usually, by an impulse map one understands a map of the form $\psi_i - id_K$.

[3] A map $r : U \to i(X)$ is a *retraction* provided that $r(y) = y$ for $y \in i(X)$.