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Solutions of the anisotropic porous medium equation in R^n under an L^1-initial value ☆

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Abstract

Consider the anisotropic porous medium equation, $u_t = \sum_{i=1}^{n}(u^{m_i})_{x_i x_i}$, where $m_i > 0$, $(i = 1, 2, \ldots, n)$ satisfying $\max_{1 \leqslant i \leqslant n}\{m_i\} \leqslant 1$, $\sum_{i=1}^{n} m_i > n - 2$, and $\max_{1 \leqslant i \leqslant n}\{m_i\} \leqslant 1/n\,(2 + \sum_{i=1}^{n} m_i)$. Assuming that the initial data belong only to $L^1(\Re^n)$, we establish the existence and uniqueness of the solution for the Cauchy problem in the space, $C([0,\infty), L^1(\Re^n)) \cap C(\Re^n \times (0,\infty)) \cap L^\infty(\Re^n \times [\varepsilon,\infty))$, where $\varepsilon > 0$ may be arbitrary. We also show a comparison principle for such solutions. Furthermore, we prove that the solution converges to zero in the space $L^\infty(\Re^n)$ as time goes to infinity.

MSC: 35K55; 35K65

Keywords: Anisotropic diffusion; Degenerate parabolic equation; Comparison principle; Large time behavior

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1. Introduction

Consider the anisotropic porous medium equation

$$u_t = \sum_{i=1}^{n} (u^{m_i})_{x_i x_i} \quad \text{in } \Re^n \times (0, \infty), \tag{1.1}$$

where m_i are positive constants. As it is well known, there have been several of works dealing with the case of all m_i's in Eq. (1.1) being the same positive constant, i.e., the case of a porous medium equation (PME). See, for example, the survey paper [2] for PME, the monograph [5] for its generalization and the references therein.

However, there are few papers on the general case of Eq. (1.1), although it has strong physical backgrounds. In fact, it follows directly when water moves in anisotropic media. If the conductivities of the media are different in different directions, the constants m_i in (1.1) must be different from each other. See [3] for details.

In papers [7,8], the first author studied the existence and uniqueness for the Cauchy problem of Eq. (1.1), provided that the initial data are bounded and continuous in $\Re^n$. He also studied the continuous modulus of solutions to (1.1) in [9] (also see Lemma 3.1 in [7]). In [6], the authors established the existence of fundamental solutions for the Cauchy problem of Eq. (1.1).

In this paper, we will study the existence, uniqueness, comparison principle and large time behavior of the solution of the Cauchy problem for (1.1) with L^1-initial data. For this purpose, we wish to consider the equation:

$$V_\tau = \sum_{i=1}^{n} [(V^{m_i})_{y_i y_i} + \alpha_i (y_i V)_{y_i}] \quad \text{in } \Re^n \times (0, \infty) \tag{1.2}$$

and its stable equation

$$-\sum_{i=1}^{n} [(f^{m_i})_{y_i y_i} + \alpha_i (y_i f)_{y_i}] = 0 \quad \text{in } \Re^n, \tag{1.3}$$

where α_i are defined by

$$\alpha_i = \frac{\overline{m} - m_i}{2} + \frac{1}{n}, \quad (i = 1, 2, \ldots, n), \quad \text{and} \quad \overline{m} = \sum_{i=1}^{n} \frac{m_i}{n}. \tag{1.4}$$

As we can see, Eq. (1.2) is equivalent to (1.1), up to a scaling transformation in spatial and time variables. See Lemma 2.2 below.

For forthcoming requirements, define

$$Q = \Re^n \times (0, \infty), \quad \beta = \overline{m} - \frac{n-2}{n} \tag{1.5}$$

and throughout this paper, we assume

$$\min_{1 \leqslant i \leqslant n} \{m_i\} \leqslant 1, \quad \beta > 0, \quad m_i > 0, \text{ and } \alpha_i > 0 \ (i = 1, 2, \ldots, n), \tag{1.6}$$