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Some new nonlinear inequalities and applications to boundary value problems

Wing-Sum Cheung*, [1]

Department of Mathematics, University of Hong Kong, Hong Kong

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Abstract

In this paper, we establish some new nonlinear integral inequalities of the Gronwall–Bellman–Ou-Iang-type in two variables. These on the one hand generalizes and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of differential equations. We illustrate this by applying our new results to certain boundary value problem.

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1. Introduction

The celebrated Gronwall–Bellman inequality [3,9] states that if u and f are non-negative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \leqslant c + \int_a^t f(s)u(s)\,\mathrm{d}s, \quad t \in [a, b],$$

* Tel.: +852 2859 1996; fax: +852 2559 2225.
E-mail address: wscheung@hku.hk.

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for some constant $c \geqslant 0$, then

$$u(t) \leqslant c \exp\left(\int_a^t f(s)\,\mathrm{d}s\right), \quad t \in [a, b]. \tag{1}$$

Since (1) provides an explicit bound to the unknown function and hence furnishes a handy tool to the study of many qualitative as well as quantitative properties of solutions of differential and integral equations, it has become one of the very few classic and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (1) have been established. Such inequalities are in general known as Gronwall–Bellman-type inequalities in the literature (see, e.g., [1,2,4–7,12–14,17,18]). Among various branches of Gronwall–Bellman-type inequalities, a very useful one is originated from Ou-Iang. In his study of the boundedness of certain second order differential equations he established the following result which is generally known as Ou-Iang's inequality.

Theorem A (*Ou-Iang [15]*). *If u and f are non-negative functions defined on* $[0, \infty)$ *such that*

$$u^2(x) \leqslant k^2 + 2\int_0^x f(s)u(s)\,\mathrm{d}s$$

for all $x \in [0, \infty)$, *where* $k \geqslant 0$ *is a constant*, *then*

$$u(x) \leqslant k + \int_0^x f(s)\,\mathrm{d}s$$

for all $x \in [0, \infty)$.

In view of the many important applications of Ou-Iang's inequality (see, e.g., [1,2,10, 13,14]), many have devoted much time and effort in its generalizations and in turn, further applications. For example, Dafermos established the following generalization of Ou-Iang's inequality in the process of establishing a connection between stability and the second law of thermodynamics.

Theorem B (*Dafermos [8]*). *If* $u \in \mathscr{L}^\infty[0, r]$ *and* $f \in \mathscr{L}^1[0, r]$ *are non-negative functions satisfying*

$$u^2(x) \leqslant M^2u^2(0) + 2\int_0^x [Nf(s)u(s) + Ku^2(s)]\,\mathrm{d}s$$

for all $x \in [0, r]$, *where* M, N, K *are non-negative constants*, *then*

$$u(r) \leqslant [Mu(0) + N\int_0^r f(s)\,\mathrm{d}s]\mathrm{e}^{Kr}.$$

More recently, Pachpatte established the following further generalizations of Theorem B.