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Multiple solutions for nonlinear operators and applications[☆]

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Abstract

In this paper, some new multiple solutions theorems are obtained by introducing some new concepts such as the O-bounded cone. The abstract results obtained here are applied to ordinary differential equations and nonlinear systems of differential equations.

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1. Introduction

In nonlinear functional analysis, the question of whether a nonlinear operator equation has multiple solutions is useful and interesting both in theory and in applications. Much attention has been given to this question by a number of authors, see [1–7,9–12]. In the 1970's, Amann [1] used the fixed point index theory to investigate the nonlinear operator equations in ordered Banach spaces and obtained the famous Amann three-solution theorem:

Theorem A *([1]). Let E be a Banach space, and P be a normal solid cone. Suppose that there exist four points $v_1, u_1, v_2, u_2 \in E$ with*

$$v_1 < u_1 < v_2 < u_2, \tag{1.1}$$

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and a completely continuous strongly increasing operator $A : E \to E$ *such that*

$$v_1 \le Av_1, \qquad Au_1 < u_1, \qquad v_2 < Av_2, \qquad Au_2 \le u_2.$$

Then A has at least three distinct fixed points in $[v_1, u_2]$.

In fact, Theorem A is still true under more general conditions. For example, it suffices to assume that

$$v_1 < u_1 < u_2, \qquad v_1 < v_2 < u_2, \qquad v_2 \not\le u_1 \tag{1.2}$$

instead of (1.1) (see [7]). However, both (1.1) and (1.2) imply that the operator A has two coupled sub–supersolutions satisfying $v_1 < v_2$ and $u_1 < u_2$. Under a more general condition, for example, in the case where A has one coupled sub–supersolution and one subsolution (or one supersolution), the number of fixed points is still unknown. In this paper, we introduce some new concepts such as the O-bounded cone to obtain some new multiple solution theorems by using the fixed point index theory under suitable conditions. Finally, the theoretical results are then applied to ordinary differential equations and nonlinear systems of differential equations, and some new existence results are obtained.

For convenience, let us introduce some definitions and symbols.

Let E be a Banach space, $P \subset E$ be a cone in E. A cone P is called solid if it contains interior points, i.e., $\overset{o}{P} \ne \emptyset$. Every cone P in E defines a partial ordering in E given by $x \le y$ iff $y - x \in P$. If $x \le y$ and $x \ne y$, we write $x < y$; if cone P is solid and $y - x \in \overset{o}{P}$, we write $x \ll y$. A cone P is said to be normal if there exists a constant $N > 0$ such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. If P is normal, then every ordered interval $[x, y] = \{z \in E \mid x \le z \le y\}$ is bounded. In this paper, the partial ordering "$\le$" is always given by P.

Let P, Q be two cones in E with $Q \subset P$. For $u \in E$, let

$$P(u) = \{x \in E \mid x - u \in P\},$$
$$Q(u) = \{x \in E \mid x - u \in Q\}.$$

Definition 1.1. A cone Q is called an O-bounded cone relative to cone P if $Q(u_0)\backslash P(v_0)$ is a bounded set for any $u_0,\ v_0 \in E$ with $v_0 \not\le u_0$.

Remark 1.1. If $v_0 \le u_0$, it is easy to see that $Q(u_0)\backslash P(v_0) = \emptyset$. So we can say that a cone Q can be called an O-bounded cone relative to cone P if $Q(u_0)\backslash P(v_0)$ is a bounded set (or an empty set) for any $u_0,\ v_0 \in E$.

From Definition 1.1, we know that $(-Q(u_0))\backslash(-P(v_0))$ is a bounded set (or an empty set) for any $u_0,\ v_0 \in E$ if Q is an O-bounded cone relative to cone P, where $-Q(u_0) = \{x \in E \mid u_0 - x \in Q\}$, $-P(v_0) = \{x \in E \mid v_0 - x \in P\}$.

Definition 1.2. Let E be a real Banach space, $P \subset E$ a solid cone, $K : E \to E$ a linear operator. Then $K : E \to E$ is said to be O-positive if there exists an O-bounded cone Q relative to cone P such that $Kx \in Q \cap \overset{o}{P}$ for $x > \theta$.

Remark 1.2. If K is an O-positive linear operator, and $x, y \in E$, $x > y$ implies $Kx - Ky \in Q \cap \overset{o}{P}$, then it is obvious that K is strongly positive, i.e., $y < x$ implies $Ky \ll Kx$.