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Opial-type inequalities for differential operators

Wing-Sum Cheung[a,*], Zhao Dandan[a], Josip Pečarić[b]

[a] *Department of Mathematics, University of Hong Kong, Hong Kong*
[b] *Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, Zagreb 10000, Croatia*

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Abstract

Some continuous and discrete versions of Opial-type inequalities which are readily applicable to differential and difference operators are established. These generalize earlier results of Anastassiou and Pečarić, and of Koliha and Pečarić.

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1. Introduction

It is well recognized that integral inequalities in general provide an effective tool for the study of quantitative as well as qualitative properties of solutions of differential and integral equations. Among these, the following Opial inequality has been one of the most useful and has continuously drawn people's attention over the past few decades.

Theorem (*Opial [17]*). *If $f \in \mathcal{L}^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0,h)$, then*

$$\int_0^h |f(x)|\,|f'(x)|\mathrm{d}x \le \frac{h}{4}\int_0^h |f'(x)|^2\mathrm{d}x.$$

Over the years, Opial's inequality has been generalized to many different situations and settings. For example, in [3,22] it was generalized to the case of many functions of one variable,

* Corresponding author. Tel.: +852 285 919 96; fax: +852 255 92 225.
E-mail addresses: wscheung@hku.hk (W.-S. Cheung), pecaric@element.hr (J. Pečarić).

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in [9–11,18,19] to the case of many functions of many variables, and in [5–7,12–14,18,21] to the cases involving higher order derivatives. Meanwhile, in [4], Agarwal first introduced and established some general Opial-type inequalities involving general ρ-derivatives, that is, for the class of differential operators $D_\rho^{(n)}$, which properly contains the class of disconjugate linear operators

$$L := D^{(n)} + \sum_{i=1}^{n} a_i(t) D^{(n-i)}.$$

Anastassiou [1] then established for the first time Opial-type inequalities for general linear differential operators, and his results were later on generalized in many directions by Anastassiou and Pečarić [2] and Koliha and Pečarić [16].

This work is a further generalization of the works in [2] and [16].

Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$, $n \in \mathbb{N}$, and we consider linear differential operators

$$L_i = D^m + \sum_{\alpha=0}^{m-1} \phi_{i,\alpha}(t) D^\alpha, \quad \phi_{i,\alpha} \in C(I), i = 1, \ldots, n.$$

Let $G_i(x, t)$ be the Green's function for L_i. For any $h \in C(I)$, it is well known (see, e.g., [15]) that for each $i = 1, \ldots, n$,

$$y_i(x) := \int_a^x G_i(x, t) h(t) \mathrm{d}t$$

is the unique solution for the initial value problem

$$\begin{cases} L_i y = h \\ y^{(j)}(a) = 0, \quad j = 0, 1, \ldots, m-1. \end{cases}$$

Collectively, if we consider $L = L_1 \otimes \cdots \otimes L_n$ as operating on $C(I) \times \cdots \times C(I)$ (n copies), then

$$\mathbf{y} := (y_1 \cdots y_n)$$

is the unique solution for the initial value problem

$$\begin{cases} L\mathbf{y} = \mathbf{h} \\ \mathbf{y}^{(j)}(a) = 0, \quad j = 0, 1, \ldots, m-1, \end{cases}$$

where $\mathbf{h} = (h, \ldots, h)$ and for each j, $\mathbf{y}^{(j)} = (y_1^{(j)}, \ldots, y_n^{(j)})$.

In [16], the case $n = 1$ was considered and estimates for the integral

$$\int_a^x U(s) |y(s)|^\beta |h(s)|^\alpha \mathrm{d}s$$

for any function $0 \le U \in C(I)$ and any real numbers $\alpha, \beta > 0$ were established. These provide valuable information on the unique solution of the aforesaid initial value problem for the case $n = 1$. In this paper, we extend the results of [16] to the case where $n > 1$, and also obtain discrete analogues which are equally useful in discrete initial value problems. As a far-reaching application, these are applied to obtain estimates of fractional derivatives.