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A posteriori stopping rule for regularized fixed point iterations☆

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Abstract

Iteratively regularized fixed-point iteration scheme

$$x_{n+1} = x_n - \alpha_n\{F(x_n) - f_\delta + \varepsilon_n(x_n - x_0)\}$$

combined with the generalized discrepancy principle

$$\|F(x_N) - f_\delta\|^2 \leqslant \tau\delta < \|F(x_n) - f_\delta\|^2, \quad 0 \leqslant n < N, \ \tau > 1,$$

for solving nonlinear operator equation $F(x) = f$ in a Hilbert space is studied in the paper. It is shown that if F is monotone and Lipschitz-continuous the sequence $\{N(\delta)\}$ is admissible, i.e.

$$\lim_{\delta \to 0} \|x_{N(\delta)} - x^*\| = 0, \tag{1}$$

where x^* is a solution to $F(x) = f$.

MSC: 47A52; 65F22

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1. Introduction

Let us consider a nonlinear operator equation

$$F(x) = f, \quad F : H \to H, \tag{1.1}$$

in a real Hilbert space H (Eq. (1.1) in a complex Hilbert space can be treated similarly). According to [6, p. 100] problem (1.1) is solvable if the operator F is monotone, hemicontinuous and $\|F(x)\| \to \infty$ as $\|x\| \to \infty$. If, in addition to that, F is strongly monotone and Lipschitz-continuous then a solution to (1.1) is unique, and it can be calculated numerically by a fixed-point iteration scheme:

$$x_{n+1} = x_n - \alpha(F(x_n) - f), \quad x_0 \in H, \tag{1.2}$$

with the appropriate choice of the parameter α. For a wide class of inverse problems in form (1.1) the operator F is not strongly monotone. In that case an iteratively regularized version of method (1.2) was proposed in [3]

$$x_{n+1} = x_n - \alpha_n\{F(x_n) - f + \varepsilon_n(x_n - x_0)\}. \tag{1.3}$$

It was shown in [3] that if (1.1) is solvable (not necessarily uniquely), F is monotone and Lipschitz-continuous then one can choose sequences $\{\alpha_n\}$ and $\{\varepsilon_n\}$ to guarantee convergence of regularized iterations (1.3) to the x_0-normal solution of (1.1), i.e. the solution nearest to x_0 in the norm of H. Suppose now that the exact right-hand side $f \in H$ in (1.1) is given by its δ-approximation:

$$\|f - f_\delta\| \leqslant \delta. \tag{1.4}$$

If f_δ does not belong to the range of F then iterates x_n in (1.3) can diverge, but still allow a stable approximation of the solution provided that the process is stopped after an appropriate number of steps $N = N(\delta)$. In this paper we suggest to choose $N(\delta)$ according to the following generalized discrepancy principle [4]:

$$\|F(x_N) - f_\delta\|^2 \leqslant \tau\delta < \|F(x_n) - f_\delta\|^2, \quad 0 \leqslant n < N, \ \tau > 1, \tag{1.5}$$

and analyze the convergence of iteratively regularized fixed-point iterations

$$x_{n+1} = x_n - \alpha_n\{F(x_n) - f_\delta + \varepsilon_n(x_n - x_0)\} \tag{1.6}$$

under the following basic assumptions:

Condition A. Problem (1.1) is solvable in H (not necessarily uniquely) and x^* is a solution. The right-hand side of (1.1) is known approximately and inequality (1.4) holds.

Condition B. The operator F is monotone in H

$$(F(h_1) - F(h_2), h_1 - h_2) \geqslant 0 \quad \text{for all } h_1, h_2 \in H, \tag{1.7}$$

and Lipschitz-continuous:

$$\|F(g_1) - F(g_2)\| \leqslant L\,\|g_1 - g_2\| \quad \text{for any } g_1, g_2 \in H. \tag{1.8}$$