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Existence results for neutral functional differential inclusions in Banach algebras

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Abstract

In this paper an existence theorem for the first-order functional differential inclusions in Banach algebras is proved under the mixed generalized Lipschitz and Carathéodory conditions. The existence of extremal solutions is also proved under certain monotonicity conditions.

MSC: 34K10

Keywords: Functional differential inclusion; Existence theorem; Extremal solutions

1. Statement of the problem

Let $\mathbb{R}$ denote the real line and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Let $J = I_0 \cup I$, then J is a closed and bounded interval in $\mathbb{R}$. Let C denote the Banach space of all continuous real-valued functions ϕ on I_0 with the supremum norm $\|\cdot\|_C$ defined by

$$\|\phi\|_C = \sup_{t \in I_0} |\phi(t)|.$$

Clearly C is a Banach algebra with this norm. For any continuous function x defined on the interval $J = [-r, a] = I_0 \cup I$ and any $t \in I$, we denote by x_t the element of C defined by

$$x_t(\theta) = x(t + \theta), \quad -r \leqslant \theta \leqslant 0, \quad 0 \leqslant t \leqslant a.$$

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Denote by $\mathscr{P}_p(\mathbb{R})$ the class of all non-empty subsets of $\mathbb{R}$ with property p. In particular, $\mathscr{P}_{\rm cl}(\mathbb{R})$, $\mathscr{P}_{\rm cv}(\mathbb{R})$ and $\mathscr{P}_{\rm cl}(\mathbb{R})$ denote respectively the classes of all closed, convex and bounded subsets of $\mathbb{R}$ respectively. Similarly $\mathscr{P}_{\rm cv,cp}(\mathbb{R})$ denote the class of all compact and convex subsets of $\mathbb{R}$.

Given a function $\phi \in C$, consider the first order neutral functional differential inclusion (in short FDI)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x(t)}{f(t,x(t),x_t)}\right] &\in G\left(t, x_t, \int_0^t k(t,s,x_s)\,\mathrm{d}s\right) \quad \text{a.e. } t \in I, \\ x(t) &= \phi(t), \ \ t \in I_0, \end{aligned} \tag{1.1}$$

where $f : I \times \mathbb{R} \times C \to \mathbb{R} - \{0\}$, $k : I \times I \times C \to \mathbb{R}$ and $G : I \times C \times \mathbb{R} \to \mathscr{P}_p(\mathbb{R})$.

Note that derivative of the denominator on the left-hand side makes the FDI (1.1) neutral, because in this case, the rate of change $x'(t)$ of the hereditary systems is determined by the values of x and x' over some past interval of time.

By a solution of FDI (1.1) we mean a function $x \in C(J,\mathbb{R}) \cap AC(I,\mathbb{R}) \cap C(I_0,\mathbb{R})$ that satisfies

(i) $t \mapsto \dfrac{x(t)}{f(t,x(t),x_t)}$ is differentiable, and

(ii) there exists a $v \in L^1(J,\mathbb{R})$ with $v(t) \in G(t, x_t, \int_0^t k(t,s,x_s)\,\mathrm{d}s)$ a.e. $t \in I$ satisfying $\frac{\mathrm{d}}{\mathrm{d}t}[\frac{x(t)}{f(t,x(t),x_t)}] = v(t)$ for all $t \in I$ and $x_0 = \phi$, where $AC(I,\mathbb{R})$ is the space of all absolutely continuous real-valued functions on I.

The functional differential inclusions have been the most active area of research since long time. See [2,11] and the references therein. But the study of functional differential equations in Banach algebra is very rare in the literature. Very recently the study along this line has been initiated via fixed point theorems. See [5,6,8] and the references therein. The FDI (1.1) is new to the literature and the study of this problem will definitely contribute immensely to the area of functional differential inclusions. Some special cases of FDI (1.1) have been studied in the literature. See [2,3] and the references therein. The fixed point theorems that will be used in the sequel for proving the main existence result is given in the following section.

2. Auxiliary results

Before stating the main fixed point theorems, we give some useful definitions and preliminaries that will be used in the sequel. Let X be a Banach space and let $\mathscr{P}(X)$ denote the class of all subsets of X. Denote

$$\mathscr{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has a property } p\}.$$

Thus $\mathscr{P}_{\rm bd}(X)$, $\mathscr{P}_{\rm cl}(X)$, $\mathscr{P}_{\rm cv}(X)$, $\mathscr{P}_{\rm cp}(X)$, $\mathscr{P}_{\rm cl,bd}(X)$, $\mathscr{P}_{\rm cp,cv}(X)$ denote the classes of all bounded, closed, convex, compact, closed-bounded and compact-convex subsets of X, respectively. Similarly $\mathscr{P}_{\rm cl,cv,bd}(X)$ and $\mathscr{P}_{\rm cp,cv}(X)$ denote the classes of closed, convex and bounded and compact, convex subsets of X, respectively. A correspondence $T : X \to$