Available online at www.sciencedirect.com
SCIENCE DIRECT®

Nonlinear Analysis 64 (2006) 856–868

Nonlinear Analysis

www.elsevier.com/locate/na

On stability of a functional equation with n variables

Jae-Hyeong Bae[a], Won-Gil Park[b,*]

[a] *Department of Mathematics and Applied Mathematics, Kyunghee University 449-701 Yongin, Republic of Korea*
[b] *Department of Mathematics, Hannam University 306-791 Daejeon, Republic of Korea*

Received 5 April 2005; accepted 9 June 2005

Abstract

Generalized Hyers–Ulam stability problems of the quadratic functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \leqslant i < j \leqslant n} f(x_i + x_j)$$

shall be treated under the approximately even (or odd) condition, and some behaviors of quadratic mappings and additive mappings shall be investigated.

MSC: 39B52; 39B72

Keywords: Ulam stability problem; Hyers–Ulam stability; Approximately even (or odd) mapping

1. Introduction

In 1940, Ulam proposed the stability problem (see [12]):

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

* Corresponding author.
E-mail addresses: jhbae@khu.ac.kr (J.-H. Bae), wgpark@math.cnu.ac.kr (W.-G. Park).

0362-546X/$ - see front matter
doi:10.1016/j.na.2005.06.028

$d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by Hyers [4] in the case of Banach space. Thereafter, we call that type the Hyers–Ulam stability. In 1978 Rassias [10] extended the Hyers–Ulam stability by considering variables instead of the constants ε and δ. It also has been generalized to the function case by Găvruta [3].

Let X and Y be a real normed space and a Banach space, respectively.

Definition. A mapping $f : X \to Y$ is called *additive* (respectively, *quadratic*) if it satisfies the equation

$$f(x + y) = f(x) + f(y)$$

(respectively, $f(x + y) + f(x - y) = 2f(x) + 2f(y)$) for all $x, y \in X$.

For a mapping $f : X \to Y$, consider the following functional equations:

$$f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(x + z) + f(y + z) \tag{1}$$

and

$$\begin{aligned} &f(x + y + z + w) + f(x) + f(y) + f(z) + f(w) \\ &\quad = f(x + y) + f(x + z) + f(x + w) + f(y + z) + f(y + w) + f(z + w) \end{aligned} \tag{2}$$

for all $x, y, z, w \in X$.

The functional equation (1) was solved by Kannappan [7]. In fact, he proved that a functional on a real vector space is a solution of Eq. (1) if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x) = B(x, x) + A(x)$ for any $x \in X$. Recently Chang, et al. [1] investigated the generalized Hyers–Ulam–Rassias stability of Eq. (2). For $n \geqslant 3$, consider the following functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + (n - 2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \leqslant i < j \leqslant n} f(x_i + x_j) \tag{3}$$

for all $x_1, \ldots, x_n \in X$. In this paper, the Hyers–Ulam stability of Eq. (3) shall be proved under the approximately even (or odd) condition.

2. Approximately even case

From now on, let $\mathbb{R}^+$ denote the set of all nonnegative real numbers.