Available online at www.sciencedirect.com
SCIENCE DIRECT®
Nonlinear Analysis 64 (2006) 499–512
Nonlinear Analysis
www.elsevier.com/locate/na

# On a nonlocal problem for nonlinear pseudoparabolic equations☆

Dao-Qing Dai*, Yu Huang

*Department of Mathematics, Sun Yat-Sen (Zhongshan) University, Guangzhou 510275, PR China*

Received 10 February 2005; accepted 21 March 2005

**Abstract**

For a nonlinear pseudoparabolic equation with one space dimension we consider its initial boundary value problem on an interval. The boundary condition on the left end is of Dirichlet type, the right end condition is replaced by a nonlocal one. Because it is given by an integral, the function involved could exhibit singularities, which distinguishes this nonlocal condition from its Dirichlet counterpart. Based on an elliptic estimate and an iteration method we established the well-posedness of solutions in a weighted Sobolev space.


*Keywords:* Nonlinear pseudoparabolic equation; Weighted Sobolev space; Elliptic estimate; Nonlocal boundary condition

## 1. Introduction

Let $[\alpha, \beta](\alpha < \beta)$ be an interval of the real line $\mathbb{R}$ and let $T$ be a positive real number. In this paper, we shall consider the following nonlinear pseudoparabolic equation with one space dimension

$$u_t - (a(x,t)u_{xt})_x = F(x,t,u,u_x,u_{xx}), \quad \alpha < x < \beta, \ \ 0 < t < T, \tag{1.1}$$

☆ This work is partly supported by NSF of China, NSF of GuangDong.

* Corresponding author.
*E-mail addresses:* stsddq@zsu.edu.cn (D.-Q. Dai), stshyu@zsu.edu.cn (Y. Huang).

0362-546X/$ - see front matter 
doi:10.1016/j.na.2005.03.112

subject to the initial condition

$$u(x,0)=u_0(x), \quad \alpha \leqslant x \leqslant \beta, \tag{1.2}$$

and the nonlocal boundary condition

$$u(\alpha,t)=0, \quad \int_\alpha^\beta u(x,t)\,\mathrm{d}x=0, \ 0\leqslant t\leqslant T. \tag{1.3}$$

The pseudoparabolic equation models a variety of physical processes, for example, long dispersive waves [2], discrepancy between the conductive and thermodynamic temperatures [6], and aggregation of populations [12], etc. Integral representations of solutions were obtained in [7,10]. Existence and uniqueness of solutions of boundary value problem of Dirichlet or Neumann type were established in [4,11,13,14]. Numerical solutions by spectral method were studied in [13]. Riemann and Riemann–Hilbert problem were investigated in [8,9].

In this paper we study the nonlocal problem. This kind of boundary conditions appears when direct measurement of physical quantities is impossible, but their average values are known. For the nonlocal boundary condition $\int_\alpha^\beta u(x,t)\,\mathrm{d}x=0$ in (1.3), because it is given by an integral, the function involved could exhibit singularities, which distinguishes this nonlocal condition from its Dirichlet counterpart $u(\beta,t)=0$.

For the nonlinear function $F$ in (1.1) we assume that

$F:[\alpha,\beta]\times[0,T]\times\mathbb{R}^3\to\mathbb{R}$ and there exists a positive constant $K$ such that

$$\begin{aligned}&|F(x,t,y_1^1,y_2^1,y_3^1)-F(x,t,y_1^2,y_2^2,y_3^2)|\\ &\quad\leqslant K(|y_1^1-y_1^2|+|y_2^1-y_2^2|+|y_3^1-y_3^2|)\end{aligned} \tag{1.4}$$

for $x\in[\alpha,\beta],\ t\in[0,T],\ y_i^j\in\mathbb{R},\ i=1,2,3,\ j=1,2.$

In [3] the nonlocal boundary condition (1.3) was studied, the equation being investigated was $u_t-(au_x)_x-(au_x)_{xt}=f(x,t,u,u_x)$ so that in our notation the nonlinear term is $f(x,t,u,u_x)+(au_x)_x+(a_tu_x)_x$ which is linear in $u_{xx}$. It was shown [3] that when the constant $K$ in (1.4) is small, there is a weak solution (in the sense that an operator is closable) to the problem (1.1)–(1.3). Using a new technique, we shall show that under the assumption (1.4), the problem has a unique solution without requiring $K$ being small. Moreover, since we are able to get estimate on the second-order derivative $u_{xx}$ the nonlinear function $F$ may have nonlinearity on $u_{xx}$.

The organization of this paper is as follows. In Section 2, for the weighted space introduced in [3] we shall derive some embedding relationship with the usual Sobolev spaces. In Section 3, we shall show existence and uniqueness of solutions of an elliptic equation in divergence form with the nonlocal boundary condition (1.3). We also show that the regularity of solutions is sharp. Finally, in Section 4, we shall establish the existence and uniqueness of solutions for problems (1.1)–(1.3). Through our new iteration method we get solutions in a weighted Sobolev space instead of in weak sense. Our method also provides a numerical scheme.

Throughout this paper, we denote by $c$ a universal constant. That means that it may change from time to time.