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Almost periodic weak solutions of neutral delay-differential equations with piecewise constant argument☆

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Abstract

In this paper, we give some theorems on the almost periodic weak solutions of second-order neutral delay-differential equations with piecewise constant argument of the form

$$(x(t)+px(t-1))''=qx\left(2\left[\frac{t+1}{2}\right]\right)+f(t),$$

where $[\cdot]$ denotes the greatest integer function, p, q are nonzero constants, and $f(t)$ is almost periodic.

Keywords: Neutral differential equation; Almost periodic; Piecewise constant argument; Weak solution

1. Introduction and preliminaries

In this paper, we consider the equation

$$(x(t)+px(t-1))''=qx\left(2\left[\frac{t+1}{2}\right]\right)+f(t), \tag{1.1}$$

where $[\cdot]$ denotes the greatest integer function, p and q are nonzero constants, and $f(t)$ is almost periodic on R. It is easy to see that (1.1) is of advanced type for $2n-1\leqslant t<2n$

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and of retarded type for $2n \leqslant t < 2n+1$, where n is an integer (see [4,8,10]). This type of functional differential equations combines the properties of differential equations and difference equations, and usually describes hybrid dynamical systems.

If $f(t) = 0$, (1.1) was considered by [5], and the asymptotic behavior of solutions was studied there. If $f(t)$ is replaced by a nonlinear function $g(t, x(t), x([t]))$, (1.1) was considered in [4,8,10] for $|p| < 1$, and some existence and uniqueness theorems of almost periodic solutions or pseudo almost periodic solutions were proved. For some other excellent works in this field we refer to [3,6,7,9] and the references therein, and for a survey of the work on differential equations with piecewise constant arguments we refer to [1].

As in [3,4,6–8], to study this sort of differential equations, we consider the corresponding difference equations, and the almost periodic (weak) solutions of the differential equations are constructed by using the almost periodic solutions of the difference equations. In fact, we will consider the following corresponding difference equations:

$$x_{n+1} + (p - 2 - q)x_n + (1 - 2p)x_{n-1} + px_{n-2} = h_n, \tag{1.2}$$

$$(1 - q/2)x_{n+1} + (p - 2)x_n + (1 - 2p - q/2)x_{n-1} + px_{n-2} = h_n. \tag{1.3}$$

Unfortunately, some strict conditions must be satisfied if we want to get the almost periodic solutions of (1.1) from the almost periodic solutions of (1.2) or (1.3) (see Theorem 2.4), and we will see also in Example 4.1 that the almost periodic solutions for (1.1) cannot be always got from (1.2) or (1.3) even if $f(t)$ is periodic and $|p| < 1$. However, some sort of weak solutions (see Definition 1.2) exist.

The main purpose of this paper is to give some existence theorems of the weak solutions of (1.1) for $|p| \neq 1$ (see Theorems 2.1 and 2.2). Meanwhile, a negative result of the uniqueness of the weak solution of (1.1) for $|p| \neq 1$ is presented (see Theorem 2.3).

The outline of this paper is as follows: the main results are presented in Section 2; Section 3 is devoted to prove Theorem 2.2; an example is given in Section 4.

Now we give some preliminary notions, definitions and propositions. Throughout this paper Z, R and C denote the sets of integers, real and complex numbers, respectively. The following Definition 1.1 can be found (or simply deduced from the theory) in any book, say [2], on almost periodic functions.

Definition 1.1. (1) A set $K \subset R$ is said to be relatively dense if there exists $L > 0$ such that $[a, a + L] \cap K \neq \emptyset$ for all $a \in R$.

(2) A bounded continuous function $f : R \to R$ is said to be almost periodic (abbreviated as a.p.) if the ε-translation set of f

$$T(f, \varepsilon) = \{\tau \in R : |f(t + \tau) - f(t)| < \varepsilon \text{ for all } t \in R\}$$

is relatively dense for each $\varepsilon > 0$.

(3) A sequence $x : Z \to R^k$ (resp. C^k), $k \in Z$, $k > 0$, denoted by $\{x_n\}$, is called an almost periodic sequence (abbreviated as a.p. sequence) if the ε-translation set of $\{x_n\}$

$$T(\{x_n\}, \varepsilon) = \{\tau \in Z : |x_{n+\tau} - x_n| < \varepsilon \text{ for all } n \in Z\}$$

is relatively dense for each $\varepsilon > 0$, here $|\cdot|$ is any convenient norm in R^k (resp. C^k). We denote the set of all such sequences $\{x_n\}$ by $APS(R^k)$ (resp. $APS(C^k)$).