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Multiplicity for semilinear elliptic equations involving singular nonlinearity☆

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1. Introduction

We investigate the number of positive solutions of the semilinear boundary value problem

$$\Delta u + f(u) = 0, \quad u > 0 \quad \text{in } B_R, \\ u|_{\partial B_R} = 0, \tag{1}$$

where $B_R \subset \mathbf{R}^n$ is the ball centered at the origin with radius R, and $f : (0, \infty) \to (0, \infty)$ is a locally Lipschitz continuous function satisfying the following conditions:

(H1) $f(u) \geqslant m > 0 \quad (u > 0)$;
(H2) there exists

$$\lim_{u \to \infty} \frac{f(u)}{u^p} \in (0, \infty) \quad \text{for some } 1 < p < 2^*,$$

where $2^* = \frac{n+2}{n-2}$ if $n > 2$ and $2^* = \infty$ if $n \leqslant 2$.
(H3)

$$\limsup_{u \to 0} u^\alpha f(u) \in [0, \infty) \quad \text{for some } 0 < \alpha < 1.$$

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The motivating example, involving singular nonlinearity, is the problem

$$\Delta u + \frac{\lambda}{u^\alpha} + u^p = 0, \quad u > 0 \quad \text{in } B_1, \qquad u|_{\partial B_1} = 0 \tag{2}$$

with some constants

$$\lambda > 0, \quad 0 < \alpha < 1 \quad \text{and} \quad 1 < p < 2^*.$$

This problem can be reduced to (1) with $f(s) = s^{-\alpha} + s^p$ using a suitable transformation.

Problems with singular nonlinearities have been studied extensively recently involving more general nonlinearities which may also depend on x. For the case of singularities in x, see [10,13] and references therein. We are mainly interested here in singularities in u. For nonlinearities where the term u^p does not appear, existence and uniqueness has been proved in [3,5,9,11,12,17]. For equations of the form $\Delta u + \frac{\lambda}{u^\alpha} - u^p$, existence and uniqueness for $p > 0$ (and also for $p = -\beta < 0$, $\beta < \alpha$) can be found in [12]. (For $p = -\beta < 0$, $\alpha < \beta < 1$ non-existence is expected [12].)

In the presence of u^p the problem exhibits a different behavior according to the sign of λ. If $\lambda < 0$, then there may be no positive solution; existence and non-existence results for the concave case $p \in [0, 1)$ can be found in [6,21,24] and generalized for x-dependent nonlinearities in [4,12]. In the latter case non-existence of positive solutions under the critical value λ^* still allows existence of non-negative solutions with a free boundary (see [4]). Related results can be found in [20] for the nonlinearity $\lambda(u^p - u^{-\alpha})$. When the parameter λ multiplies the term u^p, then existence and uniqueness holds for all λ if $u^{-\alpha}$ has a negative coefficient [12]. Here non-existence can also occur (namely for $\lambda \leqslant \overline{\lambda}$ with some $\overline{\lambda} > 0$) when $u^{-\alpha}$ has a changing-sign coefficient or when the term u^p is replaced by a bounded non-negative function $f(x)$ (see [6,12]). Conversely, a unique positive solution exists exactly for $\lambda \leqslant \overline{\lambda}$ with some $\overline{\lambda} > 0$ in two related cases when the equation contains three terms: $\Delta u + \lambda u + f(x)$ equals $u^{-\alpha}$ or $-u^{-\alpha}$. For $p = 0$ there are also some multiplicity results in [1].

For $\lambda > 0$, existence is easier to prove on a ball and has also been obtained on smooth bounded domains. For $p \in (0, 1)$ there is always a positive solution, which is also unique [12,21]. On the other hand, for $p > 1$ existence only holds for small λ [2,12]. Existence of a second positive solution has been proved for associated non-singular problems by using variational arguments. Some related results for different concave and/or convex nonlinearities can be found in [14–16,19]. For the singular case we prove here the existence of a second positive solution in the case of a ball and, moreover, that there are exactly two radial positive solutions for $n = 1$.

At the best of our knowledge, the celebrated result by Gidas et al. [8] has not been extended to the case of singular nonlinearities. Hence we cannot be sure that our problems have only radial solutions in the case of a ball, and consequently our results for the case $n > 1$ refer only to positive radial solutions.

The main results of our paper are formulated as follows.