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Existence of solutions to initial value problem for a parabolic Monge–Ampère equation and application

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Abstract

For the initial value problem of the parabolic Monge–Ampère equation $V_t V_{xx} + rx V_x V_{xx} - \theta V_x^2 = 0$, $(x, t) \in \mathbf{R} \times [0, T)$ arising from the optimal investment of mathematical finance theory, we establish the existence of solutions, whose application is also given. Here the initial function is unbounded, and a special property is required for the solution to satisfy.

MSC: 35k55; 35k15; 91b26

Keywords: Parabolic Monge–Ampère equation; Initial value problem; Unbounded initial function; Optimal portfolio

1. Introduction

From the theory of optimal investment in mathematical finance, the following initial value problem for a parabolic Monge–Ampère equation was derived in [9] (see also Appendix):

$$\begin{cases} V_t V_{xx} + rx V_x V_{xx} - \theta V_x^2 = 0, & (x, t) \in \mathbf{R} \times [0, T), \\ V(x, T) = g(x), & x \in \mathbf{R}, \\ V_{xx} < 0, & g'(x) > 0, \end{cases} \tag{1.1}$$

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where $V = V(x,t)$ is the unknown function, $\theta > 0$ and r are given constants, in a typical case, function $g(x)$ is given by the following:

$$g(x) = 1 - e^{-\lambda x}, \tag{1.2}$$

where λ is a positive constant, the practical meaning of the above parameters and functions can be found in [9, Chapters 2 and 4]. From mathematics point of view, it is the same as other parabolic Monge–Ampère equations (to see the parabolicity of the equation in (1.1), we may rewrite it as $V_t + rxV_x - \theta V_x^2/V_{xx} = 0$, which is automatically parabolic). The equation in (1.1) is also a fully nonlinear and non-uniformly parabolic equation, but the work done before on parabolic type Monge–Ampère equations all discuss the initial-boundary value problems (see [4,8,3,6] and references therein), whereas (1.1) are pure initial value problem; moreover the initial function in (1.2) is unbounded. Moreover, in order that the solution of (1.1) can be used to the corresponding problem of optimal investment, the solution is required to possess the following special property:

$$\begin{cases} V(x,t) \text{ is smooth and the function} \\ \tilde{\pi}(x,t) \overset{\text{def}}{=} -\dfrac{V_x(x,t)}{V_{xx}(x,t)}, \quad (x,t) \in \mathbf{R} \times [0,T) \\ \text{is Lipschitz continuous with respect to } x. \end{cases} \tag{1.3}$$

These characters, of course, increase the significance of the study of this problem.

For convenience we change the “terminal condition” into “the initial condition” by setting

$$V(x, T-t) = h(x,t).$$

Then (1.1) and (1.3) become, respectively,

$$\begin{cases} -h_t h_{xx} + rxh_x h_{xx} - \theta h_x^2 = 0, & (x,t) \in \mathbf{R} \times (0,T], \\ h(x,0) = g(x), & x \in \mathbf{R}, \\ h_{xx} < 0, & g'(x) > 0, \end{cases} \tag{1.4}$$

and

$$\begin{cases} h(x, T-t) = V(x,t) \text{ is smooth and the function} \\ \tilde{\pi}(x,t) \overset{\text{def}}{=} -\dfrac{V_x(x,t)}{V_{xx}(x,t)}, \quad (x,t) \in \mathbf{R} \times [0,T) \\ \text{is Lipschitz continuous with respect to } x. \end{cases} \tag{1.3$'$}$$

The main result of the paper is the following:

Theorem A. *Assume that $g(x)$ is smooth. If there exist a constant $C_1 > 0$ and $-\infty < \beta \leqslant \frac{1}{2}$ such that*

$$C_1(1+x^2)^{\beta} \leqslant -\frac{g'(x)}{g''(x)} \tag{1.5}$$

and $-g'(x)/g''(x)$ is Lipschitz continuous, then the initial value problem (1.4) *has a unique smooth solution $h(x,t)$ possessing property* (1.3′). *Hence the initial value problem* (1.1) *has a unique solution $V(x,t)$ possessing property* (1.3).