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Existence, stability and approximation of solutions for a certain class of nonlinear BVPs

Marek Galewski

Faculty of Mathematics, University of Lodz, Banacha 22, 90-238 Lodz, Poland

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Abstract

We study the existence of solutions, their stability and numerical approximations for elliptic Dirichlet problems with some general growth conditions. We consider an abstract family of equations for which we derive a new variational method to obtain existence and stability results which a further applied to concrete problems. Galerkin type approximations are also obtained.

Keywords: Duality; Variational method; Stability of solutions; Finite dimensional approximation

1. Introduction

The aim of the paper is to study a certain type of nonlinear, superlinear elliptic problems, e.g. to prove that the solution exists and that it depends continuously on a functional parameter. On the nonlinearity we impose some general growth conditions that comprise both sub- and super-linear cases. In the former case the problem of the existence of solutions may be tackled by other methods, see, e.g. [3] for an abstract approach—also applied in the present paper—but with the differential operator being the duality mapping. In the latter one, the approach may be for example by applying the mountain pass geometry with the assumption that a type of a Palais–Smale condition is satisfied. As it is well known such an assumption restricts the type of the growth of nonlinearity and also provides solutions in a weak sense, see [12]. Our method is somehow different. We assume some growth type inequality, see (1.4) that is to be satisfied at only one number d and therefore the growth of nonlinearity is not prescribed. Due to the construction of the set in which we look for the solution, we obtain the existence of a strong solution with some additional qualitative properties. The dual variational method is constructed so that to consider

E-mail address: galewski@math.uni.lodz.pl.

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both super- and sub-linear problems in a unified manner. As an example of the problem that is comprised by our method we investigate the following family of BVPs for $k \in N$

$$\begin{aligned} &- div(\varphi(y, |\nabla x(y)|)|\nabla x(y)|^{n-2}\nabla x(y)) = \nabla F_k(y, x(y)), \\ &x(y)|_{\partial\Omega} = 0, \quad x \in W_0^{1,n}(\Omega), \end{aligned} \tag{1.1}$$

where $\nabla F_k(y, x(y))$ denotes the derivative of F with respect to the second variable, $1/n+1/m=1$, under the following assumptions:

O1. $\Omega \subset R^r$ is a region with a regular boundary and $r \geqslant 2$; $n \geqslant 2$ is fixed and $n > r$, $\varphi : \Omega \times R \to R$ is a Caratheodory function, i.e. continuous with respect to x for a.e. y and measurable in y for every x; there exist constants $M_1, M_2 > 0$ such that for a.e. $y \in \Omega$ and for all $a \in R_+$,

$$M_1 \leqslant \varphi(y, a) \leqslant M_2.$$

There exists a constant $\gamma > 0$ such that for all $a \geqslant b$, $a, b \in R$ and a.e. $y \in \Omega$,

$$\varphi(y, a)a - \varphi(y, b)b \geqslant \gamma(a - b). \tag{1.2}$$

Fk1. For all $k=0, 1, 2, \ldots$ there exist a number d_k such that $\nabla F_k(\cdot, d_k) \in L^\infty(\Omega)$; and $d_k \leqslant d_0 < d$.

We define for all $k = 0, 1, 2, \ldots$ the following number:

$$g_k = \max \left\{ \sup_{y\in\Omega} esse\, |\nabla F_k(y, -d_k)|, \sup_{y\in\Omega} esse\, |\nabla F_k(y, d_k)| \right\}. \tag{1.3}$$

Fk2. $F_k, \nabla F_k : \Omega \times [-d, d] \to R$ are Caratheodory function with F_k convex in the second variable; F_k is equal $+\infty$ on the set $\Omega \times (R\backslash I)$; for a.e. $y \in \Omega$;

$$\frac{(vol(\Omega))^{1/m} c_S^{n-1}}{M_1} g_k \leqslant d_k^{n-1}, \tag{1.4}$$

where c_S is the best Sobolev constant from the inequality $1/n + 1/m = 1$

$$\max_{y\in\Omega} |x(y)| \leqslant c_S \|\nabla x\|_{L^n(\Omega)}.$$

Fk3. $\nabla F_k(y, 0) \neq 0$, for a.e. $y \in \Omega$, $y \to F(y, 0)$ is integrable.

Now each F_k is convex and l.s.c. with respect to the second variable.

The only growth type that we impose on nonlinearity is (1.4) which depends on the differential operator in that there appear a constant M_1 from its definition. With $\varphi = const$ and $M_1 = 1$, we have the classical p-Laplacian equation. It is obvious that we may also consider a Dirichlet problem with *pseudo*-Laplacian by the same method by suitable modification of the growth conditions.

We may observe by the following example what sort of growth conditions may be imposed on the problems which we consider.

Example 1.1. The sublinear case. Let n be even. Denote $\beta = ((vol(\Omega))^{1/m} c_S^{n-1})/M_1$ and consider a function $F(y, x) = [1/\beta n]x^n f(y) - [\alpha/\beta]x$ where $f \in L^\infty(\Omega)$ with $f(y) \in [\delta, \gamma]$ a.e. and α, γ, δ are positive constants. Inequality (1.4) holds for some $d > 0$ provided

$$d^{n-1}\gamma - \alpha \leqslant d^{n-1} \quad \text{and} \quad - d^{n-1}\delta - \alpha \leqslant d^{n-1}.$$

The former inequality has obviously positive solutions (with d arbitrarily large in case $\gamma < 1$) and the latter holds for any d.