Available online at www.sciencedirect.com
SCIENCE DIRECT®
Nonlinear Analysis 64 (2006) 2516 – 2526
Nonlinear Analysis
www.elsevier.com/locate/na

Local well-posedness of a new integrable equation ☆

Yongqin Liu[a,*], Weike Wang[b]

[a] School of Mathematics & Statistics, Wuhan University, 430072, China
[b] Department of Mathematics, Shanghai Jiao Tong University, 200030, China

Received 4 August 2005; accepted 31 August 2005

Abstract

In this article we focus on the local-in-time well-posedness of the Cauchy problem for a new integrable equation. We proved the local-in-time existence and uniqueness of the entropy solutions by using the method of the vanishing viscosity and L^1-contraction property.

MSC: 35G25; 35Q58

Keywords: Well-posedness; Integrable equation; Entropy solution

1. Introduction

Recently, Degasperis and Procesi studied the following family of third-order dispersive conservation laws (see [9]),

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \tag{1.1}$$

of which the right-hand side is the derivative of a quadratic differential polynomial, where α, c_0, c_1, c_2, and c_3 are real constants.

Applying the method of asymptotic integrability to the family (1.1), they find that there are only three equations that satisfy the asymptotic integrability condition within this family, namely, the KdV equation, the Camassa–Holm equation and one new equation.

☆ The research was supported in part by NNSF of China (10131050) and by Science and Technology Committee of Shanghai, China (03JC14013).

* Corresponding author.
E-mail addresses: liuyq2@hotmail.com, jarymary2@yahoo.com.cn (Y. Liu), wkwang@sjtu.edu.cn (W. Wang).

0362-546X/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2005.08.030

With $\alpha = c_2 = c_3 = 0$ in Eq. (1.1), we find the well-known Korteweg-de Vries equation which can be scaled to the following form:

$$u_t + ku_x - 6uu_x + u_{xxx} = 0 \tag{1.2}$$

which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. $u(x, t)$ represents the wave height above a flat bottom, x is proportional to the distance in the direction of propagation and t stands for the elapsed time (see [12,10]).

With $c_1 = -(3/2)(c_3/\alpha^2)$ and $c_2 = c_3/2$, we find the Camassa–Holm equation, scaled to the following form:

$$u_t - u_{txx} + c_0 u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.3}$$

which models the unidirectional propagation of shallow water over a flat bottom, $u(x, t)$ represents the fluid velocity at time $t \geqslant 0$ in the spatial x direction. The Camassa–Holm equation has a bi-Hamiltonian structure and is completely integrable (see [1,2]). Its solitary waves are smooth if $c_0 \neq 0$, and peaked in the limiting case $c_0 = 0$. Moreover the peaked solitons are stable (see [7,6]). The solutions of the equation are local well-posed with initial data $u_0 \in H^s$, $s > 3/2$. Eq. (1.3) has global strong solutions in some cases and also solutions which blow up in finite time in other cases. The weak solutions of the equation with initial data $u_0 \in H^1$ exist globally in time (see [5,3,11,4,14]).

With $c_1 = -2c_3/\alpha^2$ and $c_2 = c_3$ in Eq. (1.1), by scaling, shifting the dependent variable and applying a Galilean boost (see [8]) we find the following new equation:

$$\begin{cases} u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in R, \\ u(x, 0) = u_0(x), & x \in R \end{cases} \tag{1.4}$$

which has a form similar to the Camassa–Holm equation (1.3). However, Eq. (1.4) is truly different from Eq. (1.3). On one side, the isospectral problem for Eq. (1.4) is $\psi_x - \psi_{xxx} - \lambda y \psi = 0$ (see [8]), while the isospectral problem for (1.3) is $\psi_{xx} = (1/4)\psi + \lambda y \psi$ (see [1]). In both cases $y = u - u_{xx}$. On the other side, they have different forms of conservation laws. (1.4) has conserved quantities, $\int_R u \, \mathrm{d}x$ and $\int_R u^3 \, \mathrm{d}x$, while for (1.3), it has conserved quantities, $\int_R (u^2 + u_x^2) \, \mathrm{d}x$ and $\int_R (u^3 + uu_x^2) \, \mathrm{d}x$ (see [1,8]).

The formal integrability, the bi-Hamiltonian structure and an infinite sequence of conserved quantities of (1.4) are obtained by Degasperis–Holm–Hone [8] by constructing a Lax pair. Eq. (1.4) admits peakon solution analogous to Eq. (1.3) which are global weak solutions [8]. The existence and uniqueness of global strong solutions and global weak solutions to (1.4) with the continuous initial data has recently been studied in [15,16] where the regularity requirement on the initial data are stronger than those of the present paper, however there seems to be no papers dealing with the entropy solutions. In this paper we are interested in the well-posedness to the Cauchy problem for (1.4) with the discontinuous initial data. By means of a fixed point argument we prove the local-in-time well-posedness of the corresponding viscous equation. By using the L^1-contraction, making further estimation and choosing special test functions, we prove the local-in-time existence and uniqueness of the weak entropy solution to Eq. (1.4).

We denote the set of all the real numbers by R, and $u^+ = \max(u, 0)$. Similar to the definitions of weak, entropy solutions to the first-order conservation laws, we give the definitions of weak, entropy solution to the corresponding equation with a source term (see [13]), i.e. the following