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Nonlinear Analysis 64 (2006) 2527–2535

Nonlinear Analysis

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An approximation method for strictly pseudocontractive mappings☆

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Received 27 July 2005; accepted 31 August 2005

Abstract

Let K be a closed convex subset of a q-uniformly smooth separable Banach space, $T : K \to K$ a strictly pseudocontractive mapping, and $f : K \to K$ an L-Lispschitzian strongly pseudocontractive mapping. For any $t \in (0, 1)$, let x_t be the unique fixed point of $tf + (1-t)T$. We prove that if T has a fixed point, then $\{x_t\}$ converges to a fixed point of T as t approaches to 0.

Keywords: Strictly pseudocontractive mapping; Strongly pseudocontractive mapping; Viscosity approximation; Fixed point

Let E be a Banach space. E is said to be *smooth* if for any $x \in E$, there is a unique $x^* \in E^*$ such that $\|x^*\| = \|x\|$ and $\langle x, x^* \rangle = \|x\|^2$. We shall denote x^* by $j(x)$. (In this article, we will assume that all Banach spaces are smooth.) A Banach space E is said to be *uniformly smooth* if $\|x+y\| + \|x-y\| - 2 = o(\|y\|)$ as $\|y\| \to 0$. The function

$$\rho_E(\tau) = \sup\{\tfrac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leqslant 1, \ \ \|y\| \leqslant \tau\}$$

☆ Project supported by Colleges and Universities, Science and Technology Development Foundation (20040401) of Tianjin, PR China.

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0362-546X/$ - see front matter
doi:10.1016/j.na.2005.08.031

is called the modulus of smooth of E. Let $2 \geqslant q > 1$. E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(\tau) \leqslant c\tau^q$. It is known that

(1) for any $1 < p < \infty$, L_p and ℓ_p are $\min\{2, p\}$-uniformly smooth;
(2) every uniformly smooth space is smooth;
(3) if E is q-uniformly smooth [7], then there is a constant $c_q > 0$ such that

$$\|x + y\|^q \leqslant \|x^q\| + q\|x\|^{q-2}\langle y, j(x)\rangle + c_q\|y\|^q \quad \text{for all } x, y \in X.$$

Let X be a Banach space. The moduli of convexity of X is defined by

$$\delta(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\| : \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\}.$$

A Banach space X is said to be *p-uniformly convex*, if there is a constant $c > 0$ such that $\delta(\varepsilon) \geqslant c\varepsilon^p$ for all $\varepsilon < 1$. Xu [7] proved that if X is p-uniformly convex, then there is a continuous strictly increasing convex function $\mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leqslant \lambda\|x\|^p + (1 - \lambda)\|y\|^p - (\lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p)g(\|x - y\|)$$

for any $\lambda \in [0, 1]$ and any x, y in the unit ball of E.

Let E be a smooth Banach space and T be a mapping with domain $D(T)$ and range $R(T)$ in E. T is said to be *strongly pseudocontractive* if for any $x, y \in D(T)$,

$$\langle Tx - Ty, j(x - y)\rangle \leqslant \beta\|x - y\|^2 \quad \text{for some } 0 < \beta < 1.$$

T is said to be *strictly pseudocontractive* (in the terminology of Browder–Petryshyn) [1] if there is $\lambda > 0$ such that for any $x, y \in D(T)$,

$$\langle Tx - Ty, j(x - y)\rangle \leqslant \|x - y\|^2 - \lambda\|x - y - (Tx - Ty)\|^2. \tag{1}$$

If I denotes the identity operator, then (1) implies that for any strictly pseudocontractive mapping T,

$$\begin{aligned}&\|(I - T)x - (I - T)y\| \cdot \|x - y\| \\ &\quad \geqslant \langle (I - T)x - (I - T)y, j(x - y)\rangle \geqslant \lambda\|(I - T)x - (I - T)y\|^2,\end{aligned}$$

and

$$\|x - y\| \geqslant \lambda\|(I - T)x - (I - T)y\| \geqslant \lambda\|Tx - Ty\| - \lambda\|x - y\|. \tag{2}$$

We have proved that every strictly pseudocontractive mapping T is a Lispschitzian mapping with the Lispschitz constant $L = (\lambda + 1)/\lambda$ [5]. The following theorem shows that the fixed point set of any self strictly pseudocontractive map on a smooth space is convex.

Theorem 1. *Let K be a nonempty closed convex subset of a smooth Banach space E and let $T : K \to K$ be a strictly pseudocontractive map. Then the fixed point set $F(T)$ is a closed convex subset of E.*