Optik 126 (2015) 5163–5167

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.de/ijleo

The dual extensions of sampling and series expansion theorems for the linear canonical transform

Deyun Wei, Yuan-Min Li *

School of Mathematics and Statistics, Xidian University, No. 2, TaiBai South Street, YanTa District, Xi'an 710071, China

ARTICLE INFO

Article history:
Received 15 November 2014
Accepted 25 September 2015

Keywords:
Discrete-time linear canonical transform
Linear canonical series
Sampling
Linear canonical transform

ABSTRACT

The linear canonical transform (LCT) has been shown to be a powerful tool for optics and signal processing. In this paper, first, we present much briefer and more direct derivations of sampling relation for time-limited signal in LCT domain. Moreover, based on the sampling relation expansion in the LCT domain, the equivalent of the classical Fourier series (FS) for the LCT is derived directly. Then, the discrete-time linear canonical transform (DTLCT) is proposed through the dual extension of linear canonical series (LCS), which is the generalization of discrete-time Fourier transform (DTFT).

1. Introduction

The linear canonical transform (LCT) [1–5] is a four parameter class of linear integral transform which encompasses a number of transforms important in digital signal processing and optical system modeling. The well-known signal processing operations, such as the Fourier transform (FT), the fractional Fourier transform (FRFT), the Fresnel transform and the scaling operations are all special cases of the LCT [1–5]. Recently, along with applications of the FRFT in the signal processing community, the role of the LCT for signal processing has also been noticed. It has found many applications in optics, radar system analysis, filter design, phase retrieval, pattern recognition, time-frequency analysis, and many others applications [3–12].

The sampling process is central in almost any domain and it explains how to sample continuous signals without aliasing. The sampling theorem expansions for the LCT have been derived in [13–25], which provide the link between the continuous signals and the discrete signals, and can be used to reconstruct the original signal from their samples satisfying the Nyquist rate of that domain. However, most of sampling theorem for the band-limited signal has been derived in the LCT domain [13–21]. Moreover, a signal in a particular LCT domain was reconstructed from the samples of the signal taken in the same LCT domain. In this paper, we will present much briefer and more direct derivations of sampling relation for the time-limited signal. Specifically, a function limited at a LCT domain can be represented by its samples at any other LCT domain.

Recently, the linear canonical series (LCS) has been introduced for digital signal processing [19,26,27]. The LCS is a generalized form of FS, which can reveal the mixed time and frequency components of signals. However, it was arrived at a lengthier path [19,27]. In this paper, we will derive the equivalent of the classical Fourier series for the LCT directly based on the new sampling relation. Then, it is the important purpose of this paper to develop one scheme of linear canonical analysis method. The proposed method is the discrete-time linear canonical transform (DTLCT), and it is the generalization of discrete-time Fourier transform (DTFT) [28–31]. The DTLCT can provide a method for computing the LCT for discrete signals [28]. With the help of the algorithm of DTLCT, the LCT analysis for discrete signals can be realized.

The rest of the paper is organized as follows. In Section 2, we provide a brief review of the LCT and LCS. Sampling and series expansion theorems for LCT are derived in Section 3. The DTLCT is introduced through the dual extension of LCS in Section 4. The paper is concluded in Section 5.

2. Preliminaries

2.1. The linear canonical transform

Optical systems involving thin lenses, section of free space in the Fresnel approximation, section of quadratic graded-index media, and arbitrary combinations of any number of theses are referred to as first order optical system or quadratic-phase system.

* Corresponding author. Tel.: +86 18291835435.
E-mail addresses: dywei@xidian.edu.cn (D. Wei), ymli@xidian.edu.cn (Y.-M. Li).

http://dx.doi.org/10.1016/j.ijleo.2015.09.226

Mathematically, such system can be modeled as LCT. The LCT of a signal $f(t)$ with parameter matrix $A=(a,b;c,d)$ is defined as [1–4]

$$F_A(u)=L^A[f(t)](u)=\begin{cases}\int_{-\infty}^{\infty} f(t)K_A(u,t)\,dt, & b\neq 0,\\ \sqrt{d}e^{j(1/2)cdu^2}f(du), & b=0,\end{cases} \quad (1)$$

where

$$K_A(u,t)=\sqrt{\frac{1}{(j2\pi b)}}e^{j(1/2)[(a/b)t^2-(2/b)tu+(d/b)u^2]} \quad (2)$$

a, b, c, d are real numbers satisfying $ad-bc=1$. The inverse of the LCT is given by

$$f(t)=\int_{-\infty}^{\infty} F_A(u)K_A^*(u,t)\,du \quad (3)$$

where the * denotes complex conjugation. We only consider the case of $b\neq 0$, since the LCT is just a chirp multiplication operation if $b=0$. When $A=(\cos\theta,\ \sin\theta;\ -\sin\theta,\ \cos\theta)$, the LCT reduces to the FRFT; when $\theta=\pi/2$ it reduces to FT.

In this paper, the FT is defined as follows:

$$F(u)=\mathcal{F}(f(t))(u)=\int_{-\infty}^{\infty} f(t)e^{-jut}dt \quad (4)$$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{jut}du \quad (5)$$

where $\mathcal{F}$ denotes the FT operator.

The LCT has the following important additive, space-shift and phase-shift properties [3]

$$L^{A_2}[L^{A_1}[f(t)]]=L^{A_3}[f(t)]=F_{A_3}(u),\quad A_3=A_2\cdot A_1 \quad (6)$$

$$L^A[f(t-\tau)]=F_A(u-a\tau)e^{-jac\tau^2/2+jc\tau u} \quad (7)$$

$$L^A\left[f(t)e^{j\mu t}\right]=F_A(u-\mu b)e^{-jbd\mu^2/2+jd\mu u} \quad (8)$$

where τ, ν represent the space and phase shift parameters, respectively.

2.2. Linear canonical series

Conventional Fourier analysis has many schemes for different types of signals, such as FT and FS, which are important tools in digital signal processing [32]. The FS expansion is defined as

$$f(t)=\sum_{n=-\infty}^{\infty} C_n e^{jnt\omega_0} \quad (9)$$

$$C_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jnt\omega_0}dt \quad (10)$$

where $\omega_0=(2\pi/T)$.

The LCS is a generalized form of FS, which can reveal the mixed time and frequency components of signals. The LCS expansion of the finite-length signal $f(t)$ can be written as [19,27]

$$f(t)=\sum_{n=-\infty}^{\infty} C_{A,n}\sqrt{\frac{j}{T}}e^{-j\left\{at^2+[n(2\pi b/T)]^2 d\right\}/2b+jnt(2\pi/T)} \quad (11)$$

$$C_{A,n}=\sqrt{\frac{-j}{T}}\int_{-T/2}^{T/2} f(t)e^{j\left\{at^2+[n(2\pi b/T)]^2 d\right\}/2b-jnt(2\pi/T)}dt \quad (12)$$

where $t\in[-T/2,T/2]$, and $C_{A,n}$ are called LCS expansion coefficients with the parameter matrix A.

Lemma 1. The expansion coefficients of LCS can be obtained from the sampled values of LCT of $f(t)$

$$C_{A,n}=\sqrt{\frac{2\pi b}{T}}F_A\left(n\frac{2\pi}{T}b\right) \quad (13)$$

Proof. See [27].

3. Sampling and series expansion relations for LCT

A number of sampling and series expansion theorems for LCT have been derived [13–21,26,27]. In this section, we present much briefer and more direct and transparent derivation of sampling and series expansion relations for LCT. First, we present the dual extension of the sampling theorem with LCT for the time-limited functions. Then, we derive series expansion for LCT directly based on new sampling relation.

Sampling is one of the fundamental topics in the signal processing community. The most common sampling theorem is the Shannon sampling theorem [33] that states that if $f(t)$ is a band-limited function; it is completely expressed in terms of the time domain samples. It is possible to write the dual of Shannon's interpolation theorem for the time-limited functions. If $f(t)$ is time-limited to $[-T/2,T/2]$, the FT of $f(t)$ can be determined as

$$F(u)=\sum_{n=-\infty}^{+\infty} F(nW)\sin c\left(\frac{u}{W}-n\right) \quad (14)$$

where $W=2\pi/T$.

In the following, we develop an analogous sampling relation for LCT domain signals. Here we present a simple technique which allows briefer and more direct derivations of sampling theorem for LCT.

Based on the definition of the ordinary FT (4), the relationship between the LCT and the FT is given below:

$$\begin{aligned}F_A(u)&=\sqrt{\frac{1}{j2\pi b}}e^{jdu^2/(2b)}\int_{-\infty}^{\infty} f(t)e^{jat^2/(2b)}e^{-jut/b}dt\\ &=\left(\frac{K_A}{\sqrt{2\pi}}\right)e^{jdu^2/(2b)}\mathcal{F}\left[f(t)e^{jat^2/(2b)}\right]\left(\frac{u}{b}\right)\end{aligned} \quad (15)$$

where $K_A=\sqrt{1/(jb)}$.

According to (15), we define an intermediary function $g(t)=e^{jat^2/(2b)}f(t)$. If $f(t)$ is time-limited, so is $g(t)$. From the interpolation formula (14), we can get

$$G\left(\frac{u}{b}\right)=\sum_{n=-\infty}^{+\infty} G(nW)\sin c\left(\left(\frac{u}{bW}\right)-n\right) \quad (16)$$

By making use of this result and (15), we can derive the LCT of a time-limited function as

$$F_A(u)=\frac{K_A}{\sqrt{2\pi}}e^{j(du^2/2b)}\sum_{n=-\infty}^{+\infty} G(nW)\sin c\left(\frac{u}{bW}-n\right) \quad (17)$$