Optik 127 (2016) 10238–10246

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.de/ijleo

An analytical approach to reconstruct the initial distribution of a sub-diffusion model with time-fractional derivative☆

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ARTICLE INFO

Article history:
Received 12 June 2016
Accepted 12 August 2016

MSC:
45Q05
65M32

Keywords:
Sub-diffusion
Fractional derivative
Mittag-Leffler function
Separation of variables
Laplace transform

ABSTRACT

The inverse problem has been widely used in optical design, image processing and many other fields. This study is to employ an analytical approach to reconstruct the initial distribution of the fractional sub-diffusion model with Caputo's definition of fractional derivative in time. The basic strategy is to solve the corresponding direct problem via separation of variables and Laplace transform and then to convert the initial inverse problem into an integral equation of the first kind. The key point is that we employ the Picard's theorem to design an analytical solution of initial diffusion distribution. The proposed scheme is tested to some benchmark problems. Numerical results show that the analytical approach performs efficiently.

1. Introduction

Recent decades have witnessed an increasing application of fractional derivative in a wide range of diverse fields such as physics [1], finance [2–4], and hydrology [5,6], to mention just a few. Since 1980s, time-fractional diffusion equations have been investigated in great details by Wyss [7], Nigmatullin [8], Mainardi [9], Metzler et al. [10] and, more recently, Nigmatullin [11] and Angulo et al. [12]. These studies show that the spatial variance of the foundation solution is proportional to a fractional power of time. Later on, space-fractional diffusion equations have been investigated by Gorenflo et al. [13].

It has long been known that complex fluids cannot be accurately described by Fick's Law and Newtonian constitutive equation. Hence, kinds of fractional calculus models are considered to describe anomalous diffusion. Fomin et al. [14] employed the time-fractional differential equation to model the non-Fickian mass transport in fractured porous media. Povstenko [15] described the anomalous heat conduction in an infinite medium with a time fractional derivative. Jiang et al. [16] presented a fractional diffusion model with an absorption term and modified Fick's law to describe non-local transport processes in fractal media. It is noted that in all these studies parameters of governing fractional equations, initial and boundary conditions are assumed already a priori known, therefore, fractional calculus models can be established. However, these parameters and conditions are often unknown in the real world applications. This gives rise to essentially important inverse problems of fractional-order partial differential equations.

☆ Project supported by the National Natural Science Foundation of China (Grant Nos. 2572147 and 61472003), and the Youth Fund of Anhui University of Science and Technology (Grant No. QN201515).

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http://dx.doi.org/10.1016/j.ijleo.2016.08.032
0030-4026/

Cresson [17] investigated inverse problem of fractional calculus variations for partial differential equations to find out their Lagrangian structure. Murio [18] investigated the numerical solution of the time fractional inverse heat conduction problem (TFIHCP) on a finite slab in the presence of measured noisy data. Zhang and Xu [19] considered an inverse source problem of a fractional diffusion equation. Zhang and Wei [20] presented a new regularization method for solving a time-fractional inverse diffusion problem. Cheng et al. [21] analyze the uniqueness of an inverse problem for a one-dimensional fractional diffusion equation. Lukashchuk [22] estimated the parameters in fractional sub-diffusion equations by the time integral characteristics method. Yang [23] discussed the stability of the inverse source problem for time fractional diffusion equation. To our best knowledge, it is, however, worthy of noting that by now few work has been done on initial inverse problems of anomalous diffusion equations. In this study, we will develop an analytical technique to solve initial inverse problem of time fractional diffusion equation, which, in recent years, has been a popular model to describe anomalous diffusion such as solute transport or heat conduction in fractal porous media.

The rest of this paper is organized as follows. In Section 2, the initial inverse problem of the fractional diffusion equation is described. In Section 3, we employ an analytical approach to reconstruct the initial distribution via separation of variables and Laplace transform. In Section 4, we examine this analytical approach with some benchmark problems. In Section 5, we conclude this study with some remarks based on the results reported in this paper.

2. Initial inverse problem of fractional diffusion equation

A time fractional diffusion equation (TFDE) has been explicitly introduced in physics by Nigmatullin [8] to describe diffusion in special types of porous media which exhibit fractal geometry. In this study, we use his model to investigate one-dimensional fractional diffusion equation with Caputo's definition of time fractional derivative in a bounded domain, see Eq. (2.1), subject to the Dirichlet initial and boundary conditions (2.2) and (2.3).

$$ {}_0^C D_t^{\alpha} u(x,t) = a^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, 0 < \alpha < 1 \tag{2.1}$$

$$ u|_{t=0} = \phi(x), \tag{2.2}$$

$$ u|_{x=0} = u|_{x=1} = 0, \tag{2.3}$$

where a is a dimensionless diffusion coefficient. The bounded domain is denoted as $\Omega = \left\{ x : 0 \le x \le 1 \right\}$. For the details, see the Appendix. The Caputo's derivative definition is defined as the formula (A2).

The solution of Eq. (2.1) is unique under the initial and boundary conditions (2.2) and (2.3). For the initial inverse problem, the initial condition (2.2) is unknown and can be determined by the final distribution assumed as

$$ f(x) = u(x,T). \tag{2.4}$$

3. Methodology

In this section, we design an analytical approach to reconstruct the initial distribution of the sub-diffusion model mentioned above from the final distribution$f(x) = u(x,T)$.

3.1. Reconstruction of the initial distribution

By using the separation of variables, the solution to Eq. (2.1) is separated in terms of time and space variables by

$$ u(x,t) = X(x)V(t). \tag{3.1}$$

Substituting (3.1) into Eq. (2.1), we have

$$ \frac{{}_0^C D_t^{\alpha} V(t)}{a^2 V(t)} = \frac{X''(x)}{X(x)} = -\lambda. \tag{3.2}$$

The corresponding eigenvalue problem is given by

$$ X''(x) + \lambda X(x) = 0. \tag{3.3}$$

Equation (3.3) has nontrivial solutions if and only if the eigenvalue λ is greater than zero. Hence, the solution to Eq. (3.3) can be expresses as the following form

$$ X_n(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

where A and B are constants. From the boundary condition (2.3), we have $X(0) = X(1) = 0$, then we can obtain the solutions to Equation (3.3) in the form

$$ X_n(x) = B_n \sin n\pi x, \quad n = 1, 2, \cdots. \tag{3.4}$$