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An approximating interpolation formula for bandlimited signals in the linear canonical transform domain associated with finite nonuniformly spaced samples

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ABSTRACT

In the past literature, with the reproducing property of the reproducing kernel, Zhao et al. proposed an interpolation formula for bandlimited signals in the linear canonical transform (LCT) domain associated with finite set of irregularly spaced samples. Although the authors have discussed the reconstruction error of the formula, they did not investigate the conditions that lead to the error tends to zero when the number of samples runs to infinity, i.e., the approximating property of the formula. In this paper, we firstly present a sufficient condition achieving that property with respect to Zhao's formula. Then, under a similar condition we formulate an approximating interpolation formula for bandlimited signals in the LCT domain with parameter matrix $A=(a, b; c, d)$ associated with finite nonuniform samples of the signals' LCT with parameter matrix $\bar{A}=(-b, a; -d, c)$. We also provide some potential applications of the derived reconstruction formula to show the advantage of the theory.

1. Introduction

The linear canonical transform (LCT) is a three-free-parameter class linear integral transform [1,2] and can be considered as a generalized form of the classical Fourier transform (FT) [3], the fractional Fourier transform (FRFT) [4,5], the Fresnel transform (FST) [6], the Lorentz transform (LT) [7], and the scaling operations. The LCT has found many application fields in optics, pattern recognition, and signal processing [8–14]. Particularly, it is equivalent to a quadratic phase system (QPS), which is one of the most important optical systems and is implemented with an arbitrary number of thin lenses and propagation through free space in the Fresnel approximation or through sections of graded-index media, and therefore, it can be defined as the output light field of the QPS [9,10,14,15]

$$F_A(u)=L_A[f](u)=\begin{cases}\displaystyle\int_{-\infty}^{+\infty} f(t)K_A(t,u)\mathrm{d}t, & b\neq 0\\ \sqrt{d}e^{j(cd/2)u^2}f(du), & b=0\end{cases}, \tag{1}$$

where $f(t)$ represents the input light field, $F_A(u)$ stands for the output light field, and the LCT kernel has the form

$$K_A(t,u)=\frac{1}{\sqrt{j2\pi b}}e^{j((d/2b)u^2-(1/b)ut+(a/2b)t^2)}, \tag{2}$$

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and where the parameter matrix $A=(a, b\,; c, d)$ and the parameters a, b, c, d are real numbers satisfying $ad-bc=1$. As shown in (1), the LCT is essentially a scaling and chirp multiplication operations when $b=0$, and then it is of no particular interest to our research. Meanwhile, we can substitute the parameter matrix A with $-A$ when $b<0$. Therefore, without loss of generality, we assume $b>0$ in this paper.

With a relationship between the FT and LCT, some important theories and concepts in the FT domain have been extended to those in the LCT domain, including the sampling theorems [8,16–24]. The Shannon-type sampling theorem in the LCT domain derived in [17] states that a bandlimited signal in the LCT domain can be unique reconstructed by its infinite set of uniform samples. However, in many practical situations we expect to recover a signal from its finite nonuniform samples [18,21]. For this, Zhao et al. investigated a kind of reconstruction method for LCT-bandlimited signals associated with finite set of nonuniformly sampled points and proposed an interpolation formula interpolating the finite nonuniform samples and achieving the minimum mean-squared error (MSE) [20]. Meanwhile, reconstruction error of the derived formula was discussed, which indicates that the error decreases monotonically as the number of samples increases. Unfortunately, in general Zhao's formula does not satisfy the approximating property, that is, the error tends to zero when the number of samples runs to infinity and such a property is one of the most fundamental properties in terms of estimating the signals from finite samples. Due to this consideration, we interested in obtaining the conditions that give rise to Zhao's formula achieving the approximating property. Moreover, to the best of our knowledge, there are no results published about the approximating interpolation methods for LCT-bandlimited signals associated with finite nonuniform samples of the signals' LCT, while such a method may be particularly powerful for processing the signals corrupted with noise.

In this paper, based on the theory of functional analysis [25] we derive that Zhao's formula will achieve the approximating property when the sampling points satisfy a certain condition. Then, on the basis of a similar condition we deduce an approximating reconstruction formula for bandlimited signals in the LCT domain with parameter matrix $A=(a, b\,; c, d)$, which interpolates the finite number of nonuniformly spaced samples of the signals' LCT with parameter matrix $\bar{A}=(-b, a; -d, c)$. We also present many potential applications of the proposal.

The remainder of this paper is organized as follows. Section 2 simply reviews two important propositions in the sampling theory. In Section 3, in order to make Zhao's formula's approximating property to be satisfied, a sufficient condition related to the sampling points is obtained. Meanwhile, a newly approximating interpolation formula for LCT-bandlimited signals associated with finite set of irregularly LCTed samples is also deduced in this section. In Section 4, some possible applications of the derived results are investigated. Finally, Section 5 concludes this paper.

2. Preliminary

In this section, some essential and useful propositions in the sampling theory are given in the following.

Let H_A^σ and $L^2(-\sigma, \sigma)$ be the classes of σ-bandlimited signals in the LCT domain with parameter matrix A and square integrable functions over $(-\sigma, \sigma)$, respectively. Here, two important propositions which will be used in the later are listed as follows:

Proposition 1. *When the sampling points t_n, $n\in\mathbf{Z}$ satisfy the condition* [26]

$$|t_n-n|\le U<\frac{1}{4},\quad n\in\mathbf{Z}, \tag{3}$$

the sequence $\{e^{j(t_n\pi/\sigma)t}, n\in\mathbf{Z}\}$ forms a basis for $L^2(-\sigma, \sigma)$[27].

Proposition 2. *When the sampling points t_n, $n\in\mathbf{Z}$ satisfy the above inequality, a basis for H_A^σ can be expressed as* [19]

$$\left\{\frac{1}{\sqrt{b}}e^{-j(a/2b)t^2}\operatorname{sinc}\left(\frac{\sigma}{\pi b}\left(t-\frac{\pi b}{\sigma}t_n\right)\right), n\in\mathbf{Z}\right\}, \tag{4}$$

where $\operatorname{sinc}(t)=\sin(\pi t)/(\pi t)$.

3. Approximating interpolation formulae associated with finite nonuniformly spaced samples

In this section, we firstly review Zhao's result and obtain a sufficient condition that results in Zhao's formula achieving the approximating property. Then, with a similar condition we derive an approximating interpolation formula that interpolates the finite nonuniformly spaced samples of a LCT-bandlimited signal's LCT.

3.1. A sufficient condition that leads to Zhao's formula achieving the approximating property

In this subsection, Zhao's result is simply introduced as follows:

According to [19], the reproducing kernel of H_A^σ has a explicit expression

$$G_A(t, x)=\frac{\sigma}{\pi b}e^{j(a/2b)(x^2-t^2)}\operatorname{sinc}\left(\frac{\sigma}{\pi b}(t-x)\right). \tag{5}$$