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Original research article

Explicit solutions for space-time fractional partial differential equations in mathematical physics by a new generalized fractional Jacobi elliptic equation-based sub-equation method

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ABSTRACT

In this paper, a new generalized fractional Jacobi elliptic equation-based sub-equation method is proposed to solve space-time fractional partial differential equations in mathematical physics. This method is applied to seek exact solutions for two space-time fractional partial differential equations: the space-time fractional KP-BBM equation and the space-time fractional KdV equation. With the aid of mathematical software, a variety of exact solutions for them in the forms of the Jacobi elliptic functions are obtained.

1. Introduction

Fractional derivative is useful in describing the memory and hereditary properties of materials and processes. In general, the fractional derivative is defined in the sense of the Riemann–Liouville derivative or the Caputo derivative. Fractional differential equations containing fractional derivative are generalizations of classical differential equations of integer order. Recently, Fractional differential equations have proved to be valuable tools to the modeling of many physical phenomena, and have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, plasma, optical fibers, signal processing, systems identification, control theory, finance and fractional dynamics. To illustrate better the physical phenomena denoted by fractional differential equations, it is necessary to obtain exact solutions or numerical solutions for fractional differential equations. In the early years, due to the difficulty of obtaining exact solutions for fractional differential equations, some efficient methods have been proposed to establish approximate solutions for fractional differential equations, such as the variational iterative method [1–3], the Adomian decomposition method [4,5], the homotopy perturbation method [6–8] and so on. Recently, Jumarie proposed a modified definition for the Riemann–Liouville fractional derivative [9]. Based on the modified Riemann–Liouville derivative, many authors have investigated the methods for seeking exact solutions of fractional differential equations, and a lot of powerful methods

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have been proposed so far. These methods include the fractional sub-equation method [10–13], the (G'/G) method [14–19], the first integral method [20], the EXP method [21], the simplest equation method [22] and so on.

In this paper, by introducing a new ansätz, we propose a new fractional Jacobi elliptic equation-based sub-equation method to seek exact solutions of space-time fractional partial differential equations in mathematical physics in the sense of the modified Riemann–Liouville derivative. The main idea of this method lies in that by a traveling wave transformation $\xi=\xi(t, x_1, x_2, \ldots, x_n)$, certain fractional partial differential equation expressed in independent variables $t, x_1, x_2, \ldots, x_n$ can be turned into another fractional ordinary differential equation in ξ, the solutions of which are supposed to have the form

$$U(\xi)=a_0+\sum_{i=1}^{n} a_i G^i(\xi)+\sum_{i=2}^{n} b_i G^{i-2}(\xi)\sqrt{aG^4(\xi)+bG^2(\xi)+c}, \tag{1}$$

where the integer n can be determined by the homogeneous balancing principle, $G(\xi)$ satisfies the following fractional Jacobi elliptic equation:

$$(D_\xi^\alpha G(\xi))^2=aG^4(\xi)+bG^2(\xi)+c, \quad 0<\alpha\le 1. \tag{2}$$

Here $D^\alpha(.)$ denotes the modified Riemann–Liouville derivative of $\alpha-$ order, a, b, c are arbitrary constants. By the general solutions of Eq. (2) we can deduce the exact solutions for the original space-time fractional partial differential equation.

Remark 1. Note that if n is determined as $n=1$, then Eq. (1) reduces to $U(\xi)=a_0+a_1G(\xi)$.

We present the definition and some important properties of the modified Riemann–Liouville derivative (see [10–13]) as follows.

$$D_t^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1,\\ (f^{(n)}(t))^{(\alpha-n)}, & n\le\alpha<n+1, \quad n\ge 1.\end{cases} \tag{3}$$

$$D_t^\alpha t^r=\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}. \tag{4}$$

$$D_t^\alpha(f(t)g(t))=g(t)D_t^\alpha f(t)+f(t)D_t^\alpha g(t). \tag{5}$$

$$D_t^\alpha f[g(t)]=f'_g[g(t)]D_t^\alpha g(t)=D_g^\alpha f[g(t)](g'(t))^\alpha. \tag{6}$$

The rest of this paper is organized as follows. In Section 2, we give the description of the fractional Jacobi elliptic equation-based sub-equation method for solving space-time fractional partial differential equations. Then in Sections 3 and 4, we apply this method to seek exact solutions for the space-time fractional KP-BBM equation and the space-time fractional KdV equation respectively. In Section 5, we present some conclusions.

2. Description of the fractional Jacobi elliptic equation-based sub-equation method

In this section we give the description of the fractional Jacobi elliptic equation-based sub-equation for seeking explicit solutions of space-time fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, \ldots, x_n$, is given by

$$P(u_1,\ldots u_k, D_t^\alpha u_1,\ldots, D_t^\alpha u_k, D_{x_1}^\alpha u_1,\ldots, D_{x_1}^\alpha u_k,\ldots, D_{x_n}^\alpha u_1,\ldots, D_{x_n}^\alpha u_k, D_t^{2\alpha}u_1,\ldots, D_t^{2\alpha}u_k, D_{x_1}^{2\alpha}u_1,\ldots)=0, \tag{7}$$

where $u_i=u_i(t, x_1, x_2, \ldots, x_n)$, $i=1, \ldots, k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Suppose

$$u_i(t, x_1, x_2, \ldots, x_n)=U_i(\xi), \quad i=1,\ldots,k,$$

and a traveling wave transformation

$$\xi=lt+k_1x_1+k_2x_2+\cdots+k_nx_n+\xi_0. \tag{8}$$

Then by the second equality of (6), Eq. (7) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\widetilde{P}(U_1,\ldots,U_k, l^\alpha D_\xi^\alpha U_1,\ldots, l^\alpha D_\xi^\alpha U_k, k_1^\alpha D_\xi^\alpha U_1,\ldots, k_1^\alpha D_\xi^\alpha U_k,\ldots, k_n^\alpha D_\xi^\alpha U_1,\ldots, k_n^\alpha D_\xi^\alpha U_k, c^{2\alpha}D_\xi^{2\alpha}U_1,\ldots, c^{2\alpha}D_\xi^{2\alpha}U_k, k_1^{2\alpha}D_\xi^{2\alpha}U_1,\ldots)=0. \tag{9}$$

Step 2. Suppose that the solution of (9) can be expressed by a polynomial in $G(\xi)$ as follows:

$$U_j(\xi)=a_0+\sum_{i=1}^{m_j} a_{j,i}G^i(\xi)+\sum_{i=2}^{m_j} b_{j,i}G^{i-2}(\xi)\sqrt{aG^4(\xi)+bG^2(\xi)+c}, \quad j=1,2,\ldots,k, \tag{10}$$