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The wave packet transform associated with the linear canonical transform

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ABSTRACT

Linear canonical transforms (LCTs) are a family of integral transforms with wide application in optical, acoustical, electromagnetic, and other wave propagation problems. In this paper, a new kind of wave packet transform (WPT) associated with the LCT is proposed, this new WPT (WPTL) is defined based on the ideal of the LCT and the WPT. Some properties and physical meaning of the WPTL are investigated. In particular, we show a version of the resolution of the identity of WPTL. Moreover, the relationship between the WPTL and the Wigner distribution (WD) is derived. At last, we introduce the concept of the fractional wavepacketgram, which is defined as the modulus square of the WPTL. It is proved that the fractional wavepacketgram is a member of the Cohen class time–frequency distribution where the kernel is a scale dependent ambiguity function.

1. Introduction

It is well known that the Fourier transform (FT) is an important tool for processing stationary signals [1]. However, the actual signals are often time-varying or non-stationary. For time-varying signals, the short-time Fourier transform (STFT) is employed and it typically uses a time window of fixed length applied at regular intervals to obtain a portion of the signal assumed to be stationary [2]. The resulting time-varying spectral representation is critical for non-stationary signal analysis, but in this case it comes at fixed spectral and temporal resolution. Wavelet analysis presents an attractive alternative to the STFT by utilizing windows of variable width, which can effectively provide resolution of varying granularity [3]. In the past decades, wavelet packet transform (WPT) has been gained much attention [4–13] and successfully applied in de-noising, image compression and encryption in wireless communication [9–11], The reason of its wide usage in signal processing is that it has some better virtues than wavelet transform (WT), as it can realize multilevel decomposition and analyze the high frequency that is not achieved in common discrete WT. According to the characteristics of a signal, we choose the corresponding frequency subbands by wavelet packet decomposition and improve the time–frequency resolving ability.

The classical WPT is related with FT. The WPT is the FT of a signal windowed with a wavelet. As a generalization of the FT, the FRFT has been widely developed in theory and in application [14–17]. Recently, the linear canonical transform (LCT) as the generalization of the FT and FRFT was introduced during the 1970s with four parameters, has been proven to be one of the most powerful tools for non-stationary signal processing [17–19]. Now, it has been applied for filter design, time–frequency analysis, signal reconstruction, radar system analysis and many others [20–29].

As one of the generalization of the classical WPT, the fractional WPT have been introduced [11–13]. Also, several simple property and applications for fractional WPT have been discussed. Although potentially useful for signal processing applications, this transform appears to have remained largely unknown to the signal processing community. Especially, the relationship of fractional WPT with other time–frequency representations such as Wigner distribution (WD), the ambiguity function and the spectrogram has not been studied. In this paper, the WPT based on the LCT (WPTL) is derived to solve this problem and to present some new relationships. We first propose the new kind of WPT (WPTL), combining the idea of the LCT and WPT. Then, we obtain a version of the resolution of the identity and some properties of WPTL connected with those of LCT and WPT. In addition, we present a simple and natural relationship between the WPTL and the WD. At last, the fractional wavepacketgram is introduced and analyzed.

The rest of the paper is organized as follows. In Section 2, we provide a brief review of the WPT and LCT. In Section 3, the new definition of the WPT associated with LCT is proposed, and some

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properties are discussed. The fractional wavepacketgram is defined and discussed in Section 4. The paper is concluded in Section 5.

2. Preliminaries

2.1. Wave packet transform (WPT)

The short-time Fourier transform (STFT) is the most widely used method in signal processing for studying non-stationary signals. The continuous STFT of a signal $f(t)$ is defined as [2]

$$\mathrm{STFT}_f(\tau,u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}g(t-\tau)f(t)e^{-\mathrm{j}ut}\mathrm{d}t \tag{1}$$

where $g(t)$ is the window function. Wavelet analysis presents an attractive alternative to the STFT by utilizing windows of variable width, which can effectively provide resolution of varying granularity. The continuous wavelet transform (CWT) of a signal is defined as [3]

$$\mathrm{CWT}_f\left(\alpha,\beta\right)=\frac{1}{\sqrt{\alpha}}\int_{-\infty}^{+\infty}f(t)\psi\left(\frac{t-\beta}{\alpha}\right)\mathrm{d}t \tag{2}$$

where t is time, β is the translation parameter, α is the scale parameter and $\psi(t)$ is the transforming function, called mother wavelet. Here $\alpha>0$ and ψ is normalized such that the L^2 norm $\left\|\psi\right\|=1$.

The WPT is the combination of STFT and CWT, and it is described as [4–7]

$$\mathrm{WPT}_f\left(u,\alpha,\beta\right)=\frac{1}{\sqrt{2\pi\alpha}}\int_{-\infty}^{+\infty}f(t)\psi\left(\frac{t-\beta}{\alpha}\right)e^{-\mathrm{j}ut}\mathrm{d}t \tag{3}$$

In other words, the WPT is the FT of a signal windowed with a wavelet that is dilated by α and translated by β.

2.2. Linear canonical transform (LCT)

The LCT provides a mathematical model of paraxial propagation though quadratic phase systems. The LCT of a signal $f(t)$ with parameter matrix $A=(a,b;c,d)$ is defined as [17–19]

$$F_f^A(u)=L^A[f(t)](u)=\begin{cases}\displaystyle\int_{-\infty}^{\infty}f(t)K_A(u,t)\mathrm{d}t, & b\neq 0,\\ \sqrt{\mathrm{d}}e^{j1/2cdu^2}f(\mathrm{d}u), & b=0,\end{cases} \tag{4}$$

where the kernel function

$$K_A(u,t)=\sqrt{\frac{1}{(j2\pi b)e}}^{\,j1/2[(a/b)t^2-(2/b)tu+(d/b)u^2]} \tag{5}$$

where a, b, c, d are real numbers satisfying $ad-bc=1$ and L^A is the unitary LCT operator. The kernel has the following properties [17,27] which will be useful in this paper.

$$\overline{K_A}(u,t)=K_{A^{-1}}(t,u) \tag{6}$$

$$\int_{-\infty}^{+\infty}K_{A_2}(u,t)K_{A_1}(t,v)\mathrm{d}u=K_{A_1A_2}(u,v) \tag{7}$$

$$\int_{-\infty}^{+\infty}K_A(u,t)\overline{K_A}\left(u',t\right)\mathrm{d}u=\delta\left(u-u'\right) \tag{8}$$

where the overbar indicates complex conjugate. These two properties (7) and (8) actually correspond to the additivity property and the reversibility property of LCT. We only consider the case of $b\neq 0$, since the LCT is just a chirp multiplication operation if $b=0$. When $A=\left(\cos\theta,\sin\theta;-\sin\theta,\cos\theta\right)$, the LCT reduces to the FRFT; when $\theta=\pi/2$ it reduces to FT. For more properties and the relations with other transforms about LCT, one can refer to [17–19].

3. Wave packet transform associated with the linear canonical transform

In this section, we propose a new definition of the WPT associated with LCT based on the product of the signal and a local kernel function. Then, some properties and physical meaning of the newly defined WPT (WPTL) discussed, the results show that this kind of WPT can be seen as one generalization of the classical WPT. In addition, a very simple and natural relationship between the WPTL and the WD is presented.

3.1. The new definition of WPT associated with the LCT

Definition 1. The WPT of a signal $f(t)$ associated with the LCT (WPTL) with parameter $A=(a,b;c,d)$ is defined as

$$\begin{aligned}F_f^A\left(u,\alpha,\beta\right)&=WPTL[f(t)]\left(u,\alpha,\beta\right)\\&=\int_{-\infty}^{+\infty}\psi_f\left(t,\alpha,\beta\right)K_A(u,t)\mathrm{d}t\\&=\sqrt{1/(j2\pi\alpha b)}e^{j(d/2b)u^2}\\&\quad\int_{-\infty}^{+\infty}\psi\left(\left(t-\beta\right)/\alpha\right)f(t)e^{j(a/2b)t^2}e^{-j(1/b)ut}\mathrm{d}t\end{aligned} \tag{9}$$

where $\psi_f\left(t,\alpha,\beta\right)=1/\sqrt{\alpha}\psi\left(\left(t-\beta\right)/\alpha\right)f(t)$ and $K_A(u,t)$ is given by (5).

From the definition and the physical meaning of the LCT [17–19], the WPTL can be interpreted as the affine transform of the signal $\psi_f\left(t,\alpha,\beta\right)$ in the $\left(u,\alpha,\beta\right)$ plane. Noted that the WPTL F_f^A, is a function of time, frequency and scale.

Obviously, when $A=(0,1;-1,0)$, the LCT reduces to FT. Accordingly, the WPTL reduces to classical WPT. From the definition of WPTL (9), we see that the computation of the WPTL corresponds to the following steps:

(1) a product by a wavelet
(2) a product by a chirp
(3) a Fourier transform (with its argument scaled by $1/b$)
(4) another product by a chirp
(5) a product by a complex amplitude factor.

3.2. Some properties of WPTL

In this subsection, some properties of WPTL are investigated and the proof of some complex properties will be given in detail. Similarly to the LCT, space shift and phase shift properties with parameter $A=(a,b;c,d)$ are derived as follows:

$$F_{f(t-t_0)}^A\left(u,\alpha,\beta\right)=e^{-jact_0^2/2+jcut_0}F_f^A\left(u-at_0,\alpha,\beta+t_0\right) \tag{10}$$

$$F_{f(t)e^{j\mu t}}^A\left(u,\alpha,\beta\right)=e^{-jbd\mu^2/2+jd\mu u}F_f^A\left(u-b\mu,\alpha,\beta\right) \tag{11}$$