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On cylindrical model of electrostatic potential in fractional dimensional space

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ABSTRACT

The Laplace's equation in a fractional dimensional space describes the electrostatic potential inside fractal media in the framework of fractal continuum models. An exact solution of the Laplace's equation for cylindrical coordinate system in a space having fractional (non-integer) dimensions is derived and discussed. The discussion is divided into different cases. These cases are based on the values of parameters describing the order of fractional dimensional space and the parameter used to describe azimuthal and/or radial dependency of the potential. A coaxial cable with constant potential and filled with a fractional dimensional space is also solved as an example.

1. Introduction

The concept of fractional dimensional space is effectively used in many areas of physics to describe the effective parameters of physical systems [1–3]. This approach suggests to replace the real confining structure with an effective space, where the measurement of its confinement is given by non-integer dimension. There has been an increasing interest to study electrodynamics based on continuum models of fractal distribution of charges, currents and fields by solving various electromagnetic equations in the framework of fractional dimensional spaces [4–9]. The fractional (non-integer) dimensional solutions of some electromagnetic equations have been provided by Muslih et al. [10–12], by Hira et al. [13–21], and by Balankin et al. [22].

In electromagnetics, situations when operating frequency becomes zero is known as electrostatics whereas quasi-statics assumes that size of the object is very small compared to wavelength of the operating frequency. The problems related to electrostatics and quasi-statics are usually treated using the Laplace's equation, i.e., $\nabla^2\Phi = 0$. In three dimensional space, the expression for Laplacian operator for cylindrical coordinate system is

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \quad (1)$$

The solutions of the Laplace's equation are available in all undergrade text books on electromagnetics. The form of solutions for the scalar potential depends on the coordinate system being used. Using method of separation of variables, the solution of the Laplace's equation in cylindrical coordinates (ρ, ϕ) is given below [23]

$$\Phi_m(\rho,\phi) = \rho^m(A_m \cos m\phi + B_m \sin m\phi) + \rho^{-m}(A'_m \cos m\phi + B'_m \sin m\phi), \quad m = 1, 2\ldots \quad (2)$$

where m in above expression can be integer or non-integer. For $m = 0$ and azimuthal invariance, the potential is given below

$$\Phi(\phi,\rho) = \Phi(\rho) = C_1 + C_2 \ln \rho \quad (3)$$

For $m = 0$ and radial invariance, the potential becomes

$$\Phi(\phi,\rho) = \Phi(\phi) = C_1 + C_2\phi \quad (4)$$

This solution may be divided into sub-categories based on value of m. For example: both infinite circular cylinder and infinite wedge are analyzed in cylindrical coordinate system but potential in the case of circular cylinder is azimuthally periodic whereas potential in case of wedge is azimuthally bounded. The boundedness in case of wedge and periodicity in case of cylinder is managed by taking value of m as non-integer and integer, respectively.

For fractional space, the Laplacian operator in Cartesian coordinate system is given below [19]

$$\nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z} \quad (5)$$

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where $D=\alpha_1+\alpha_2+\alpha_3$. α_1, α_2 and α_3 can be non-integer. It may be noted that $\alpha_1=\alpha_2=\alpha_3=1$ means classical situation. Whereas in cylindrical coordinates, corresponding expression for operator ∇_D^2 is [19]

$$\nabla_D^2 = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}(\alpha_1+\alpha_2-1)\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\left(\frac{\partial^2}{\partial\phi^2}-\{(\alpha_1-1)\tan\phi-(\alpha_2-1)\cot\phi\}\frac{\partial}{\partial\phi}\right) + \frac{\partial^2}{\partial z^2}+\frac{\alpha_3-1}{z}\frac{\partial}{\partial z} \tag{6}$$

Using method of separation of variables, solutions of the scalar Helmholtz's equation in non-integer dimensional space for cartesian, cylindrical and spherical coordinate systems were derived by Zubair et al. [15–18]. It was shown that, in cartesian coordinate system, special function models the propagation of wave in direction which has non-integer dimension. Their treatment also shows that, for general solution in cylindrical coordinates, hypergeometric functions model the azimuthal dependency whereas radial dependency is model through Bessel and Hankel functions. In order to treat the static and quasi-static problems, solution of the Laplace's equation is required. This is because solution of the Helmholtz's equation cannot easily be reduced to corresponding solution derived through Laplace's equation by the limiting procedure. It may also be noted that the solution of one dimensional Laplace and Poisson's equation is available in [19].

In this paper, solution of the Laplace's equation for two dimensional boundary value problems when the host medium is of fractional dimensional space is derived. Effects on behavior of the potential due to values of parameters (α_1, α_2) describing the order of the fractional space and parameter m describing the azimuthal and/or radial dependence of potential are discussed. For this purpose, discussion is divided into four cases as follows

- Case 1: $\alpha_1 \neq 1, \alpha_2 \neq 1, m \neq 0$.
- Case 2: $\alpha_1 \neq 1, \alpha_2=1, m \neq 0$.
- Case 3: $\alpha_1 \neq 1, \alpha_2=1, m=0$.
- Case 4: $\alpha_1=1, \alpha_2=1, m=0$.

The purpose of this discussion is to derive solution for the above cases and to highlight their difference from corresponding solution of Helmholtz's equation. It may be noted that Case 4 is available in all undergrade text books on electromagnetics [see e.g., [23]). Coaxial cable with constant potentials and wedge with constant potentials are the two examples for Case 4. That is, solution is only function of radial dependency for coaxial cable whereas only function of azimuthal dependency for wedge. Case 3 is conversion of Case 4 to non-integer dimension. Case 2 and Case 1 are about situations when solution is function of both radial and azimuthal coordinates. A coaxial cable of constant potential and filled with fractional space is treated as an example.

2. Exact solution of cylindrical Laplace's equation in fractional spaces

For two dimensional potential $\Phi(\phi, \rho)$, Laplace's equation in the fractional space is given below

$$\nabla_D^2\Phi(\rho,\phi)=0 \tag{7}$$

According to the method of separation of variables, potential function $\Phi(\rho, \phi)$ may be written as product of two functions each of which depends only on one variable, i.e.,

$$\Phi(\rho,\phi)=f(\rho)g(\phi) \tag{8}$$

This reduces partial differential equation given in (7) into the following two ordinary differential equations

$$\left[\rho^2\frac{d^2}{d\rho^2}+\rho(\alpha_1+\alpha_2-1)\frac{d}{d\rho}-m^2\right]f(\rho)=0 \tag{9}$$

$$\left[\frac{d^2}{d\phi^2}-\{(\alpha_1-1)\tan\phi-(\alpha_2-1)\cot\phi\}\frac{d}{d\phi}+m^2\right]g(\phi)=0 \tag{10}$$

where parameter m is integer if $g(\phi)$ is periodic in ϕ while m is non-integer if range of ϕ is restricted.

If we set $\xi=\sin^2\phi$, the derivatives are obtained as given below

$$\begin{aligned}
d\xi &= 2\sin\phi\cos\phi \\
d\xi^2 &= 2(1-2\sin^2\phi)d\phi^2 = 2(1-2\xi)d\phi^2 \\
\frac{\tan\phi}{d\phi} &= \frac{2\sin^2\phi}{d\xi} = \frac{2\xi}{d\xi} \\
\frac{\cot\phi}{d\phi} &= \frac{2(1-\xi)}{d\xi}
\end{aligned}$$

2.1. Analysis for ϕ dependency

For ϕ dependency, Eq. (10) takes the following form

$$\left[(2\xi-1)\frac{d^2}{d\xi^2}-\left\{(2-\alpha_1-\alpha_2)\xi+(\alpha_2-1)\right\}\frac{d}{d\xi}-\frac{m^2}{2}\right]g(\phi)=0 \tag{11}$$

Comparison of the above differential equation with (B1) given in Appendix B yields

$$\begin{aligned}
a_0 &= 0 \\
b_0 &= -\frac{m^2}{2} \\
a_1 &= -(2-\alpha_1-\alpha_2) \\
b_1 &= -(\alpha_2-1) \\
a_2 &= 2 \\
b_2 &= -1 \\
x &= \xi
\end{aligned}$$

It is obvious that for $\alpha_1 \neq 1$ and/or $\alpha_2 \neq 1$, following restriction holds

$$a_2 \neq 0, \quad a_1^2 \neq 4a_0a_2.$$

So, the general solution of (11), under the given restriction, is given below

$$g(\phi)=e^{k\xi}w(z)=e^{k\xi}[C_1\Phi(a,b,z)+C_2z^{1-b}\Phi(a-b+1,2-b;z)] \tag{12}$$