Optik 127 (2016) 5255–5258

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.de/ijleo

A semi-analytical solution of Hunter–Saxton equation

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ARTICLE INFO

Article history:
Received 19 January 2016
Accepted 29 February 2016

Keywords:
Haar wavelets
Hunter–Saxton equation
Adomian decomposition method
Quasilinearization technique
Analytical solution

ABSTRACT

In this paper, we discretize time derivative terms by a forward difference scheme and linearize the nonlinear terms using a quasilinearization technique to reduce the original equation into a system of ordinary differential equations. Then the Haar wavelet quasilinearization approach is applied to compute the numerical solutions of the Hunter–Saxton equation. Computer simulations show that our obtained results are in a good agreement with the exact solution.

1. Introduction

Hunter–Saxton equation [1],

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \qquad (1)$$

or equivalently,

$$u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 = 0, \qquad (2)$$

where x and t are scaled position and time coordinates respectively and the initial condition is as follows

$$u(x, 0) = f(x), \qquad (3)$$

with boundary condition

$$\lim_{x\to\infty} u(x, 0) = 0. \qquad (4)$$

For other situations, it may be more reasonable to enforce a boundary condition

$$\lim_{t\to\infty} u(x, 0) = 0, \qquad (5)$$

which yields a solution that decays for large time. The Hunter–Saxton (HS) equation is a nonlinear wave equation which has been used to describe waves in a massive director field of a nematic liquid crystal and arises as the short-wave limit of the Camassa–Holm equation [2], an integrable model the unidirectional propagation of shallow water waves over a flat bottom [3]. It has a re-expression of the geodesic flow on the diffeomorphism group of the circle with a bi-Hamiltonian structure [4] which is completely integrable [5]. In literature, many numerical methods have been proposed for approximating solution of the generalized Burgers–Huxley equation. Baxterq et al. [6] obtained the Separable Solutions and Self-Similar Solutions of the Hunter–Saxton wave equation. Wei and Yin also studied the periodic Hunter–Saxton equation with weak dissipation [7]. Yin [8] proved the local existence of strong solutions of the periodic Hunter–Saxton equation and showed that all strong solutions except space-independent solutions blow up in nite time [9]. Wei obtained global weak solution for a periodic generalized Hunter–Saxton equation in [9].

Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of partial differential equations so far. For example, the sine-cosine method [10], the He's semi-inverse method [11], the wavelet spectral analysis [12], the direct algebraic method [13], the extended tanh-function method [14], the homotopy analysis method [15] and the extended homoclinic test approach [16].

In recent years the wavelets have been used for the solution of partial differential equations. Different types of wavelets and approximating functions have been used in numerical solution of differential equations. The Haar wavelet quasilinearization approach is the simplest one among the different wavelet families which are defined by an analytical expression. Due to its simplicity, Haar wavelet quasilinearization approach is very effective to solve ordinary and partial differential equations. The notion of wavelets is introduced by Alfred Haar in 1910. Expression of Haar wavelets

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http://dx.doi.org/10.1016/j.ijleo.2016.02.065
0030-4026/

is very simple. Besides they have orthogonal and normalization with compact support properties. Therefore, the Haar wavelets are very efficient and effective tools to solve the nonlinear systems in physics, biology, chemical reactions and fluid mechanics [17–23]. In this paper, we will apply the Haar wavelet quasilinearization approach for solving Hunter–Saxton equation.

This paper is organized as follows: in the next section we present a brief introduction on preliminaries of Haar wavelets and its integrals. Section 3 describes the quasilinearization technique for nonlinear terms. Convergence of method is discussed in Section 4. In Section 5, Haar wavelets method is used for solving the Hunter–Saxton equation. Numerical results are presented in Section 6 which Eq. (1) with the initial conditions and boundary conditions is solved. Finally, the paper in concluded in Section 7.

2. Haar wavelets

The Haar wavelets family $\{h_i(x)\}$ is defined as a group of orthogonal square waves with magnitude ± 1 in some intervals and zero elsewhere as follows

$$h_i(x) = \begin{cases} 1 & x \in [\xi_1, \xi_2), \\ -1 & x \in [\xi_2, \xi_3), \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

where the notations $\xi_1 = k/m$, $\xi_2 = (k+0.5)/m$, $\xi_3 = (k+1)/m$ are introduced. The integer $m = 2^j$, $(j = 0, 1, \ldots, J)$ indicates the level of the wavelets, where J is the maximal level of resolution and $k = 0, 1, \ldots, m-1$ is the translation parameter. The subscript i can be expressed as $i = m + k + 1$, such that in the case of $m = 1$, $k = 0$ we have $i = 2$; the maximal value of i is $i = 2M = 2^{J+1}$. For $i = 1$, the function $h_1(x)$ is a scaling function for the family of the Haar wavelets as

$$h_1(x) = \begin{cases} 1 & x \in [0, 1), \\ 0 & \text{otherwise}. \end{cases}$$

We introduce the following notations; for $i > 1$

$$P_{i,1}(x) = \int_0^{x_l} h_i(x)dx, \tag{7}$$

$$P_{i,n+1}(x) = \int_0^{x} P_{i,n}(x)dx, \quad n = 1, 2, \ldots \tag{8}$$

where for $l = 1, 2, \ldots, 2M$, $x_l = (l-0.5)/2M$ are the collocation points. By using Eqs. (6) and (8) we have

$$P_{i,n}(x) = \begin{cases} 0 & x < \xi_1, \\ \dfrac{(x-\xi_1)^n}{n!} & x \in [\xi_1, \xi_2), \\ \dfrac{(x-\xi_1)^n}{n!} - 2\dfrac{(x-\xi_2)^n}{n!} & x \in [\xi_2, \xi_3), \\ \dfrac{(x-\xi_1)^n}{n!} - 2\dfrac{(x-\xi_2)^n}{n!} + \dfrac{(x-\xi_3)^n}{n!} & x > \xi_3. \end{cases} \tag{9}$$

It is well-known that any integrable function $u(x) \in L^2[0, 1)$ can be expanded by a Haar series with an infinite number of terms as follows

$$u(x) = \sum_{i=1}^{\infty} a_i h_i(x). \tag{10}$$

The above series terminates at finite terms if $u(x)$ is a piecewise constant or can be approximated as a piecewise constant function during each subinterval, then $u(x)$ will be terminated at finite terms, i.e.,

$$u(x) = \sum_{i=1}^{2M} a_i h_i(x) = a^T_{(2M)} h_{(2M)}(x), \tag{11}$$

where the coefficients $a^T_{(2M)}$ and the Haar function vector $h_{(2M)}(x)$ are defined as

$$a^T_{(2M)} = [a_1, a_2, \ldots, a_{2M}]$$

$$h_{(2M)}(x) = [h_1(x), h_2(x), \ldots, h_{2M}(x)]^T,$$

where superscript T shows the transpose operator and $M = 2^J$.

3. Quasilinearization

The quasilinearization approach [23] is a generalized Newton–Raphson technique for functional equations. It converges quadratically to the exact solution. Also, if there is a convergence at all, it has a monotonic convergence.

Consider the nonlinear nth order differential equation

$$L^n u(x) = f(u(x), u'(x), \ldots, u^{n-1}(x), x). \tag{12}$$

Application of quasilinearization technique to (12) yields

$$L^n u_{r+1}(x) = f(u_r(x), u'_r(x), \ldots, u_r^{n-1}(x), x) + \sum_{j=0}^{n-1} (u^j_{r+1}(x) - u^j_r(x)) f_{u^j}(u_r(x), u'_r(x), \ldots, u_r^{n-1}(x), x), \tag{13}$$

with the initial and boundary conditions at $(r+1)$th iteration, where n is the order of the differential equation. Eq. (13) is always a linear differential equation and can be solved recursively, where $u_r(x)$ is known and one can use it to get $u_{r+1}(x)$.

4. Convergence of Haar wavelet method

Multi-resolution analysis (MRA) is the best way to understand the notion of wavelets [24]. Let $u(x) \in L^2[0, 1)$, MRA of $L^2[0, 1)$ generates a sequence of subspaces $V_j, V_{j+1}, V_{j+2}, \ldots$ of $L^2[0, 1)$ in such a way that the projection of $u(x)$ onto these spaces produces more magnificent approximations of the function $u(x)$ as $J \to \infty$, then the corresponding error at Jth level may be defined as

$$e_J(t) = |u(t) - u_J(t)| = |u(t) - \sum_{i=1}^{2^{J+1}} a_i h_i(t)| = |\sum_{i=2^{J+1}}^{\infty} a_i h_i(t)|. \tag{14}$$

We can analyze the error for nonlinear partial differential equations. Convergence of the method may be discussed on the same lines as given in Saeedi et al. [25]. We can also discuss the convergence of the method for nonlinear partial differential equations if we know the exact solution.

Theorem. *Suppose that $f(x)$ satisfies a Lipschitz condition on $[0, 1]$, that is, there exists positive K such that for all $x, y \in [0, 1]$ we have $|f(x) - f(y)| \le K|x - y|$, K is the Lipschitz constant. The error bound for $\|e_J(x)\|_2$ is also obtained as*

$$\|e_J(x)\|_2 \le \left(\frac{M}{\sqrt{3}2^{J+1}} \right). \tag{15}$$