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New convolution structure for the linear canonical transform and its application in filter design

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ABSTRACT

The convolution and product theorem for the Fourier transform (FT) plays an important role in signal processing theory and application. The linear canonical transform (LCT), which is a generalization of the FT and the fractional Fourier transform (FRFT), has found many applications in optics and non-stationary signal processing. Recently, some scholars have formulated a series of convolution and product theorems for the LCT, however, both of them do not maintain the convolution theorem for the FT. The purpose of this paper is to present a new convolution structure for the LCT having the elegance and simplicity in both time and LCT domains comparable to that of the FT. We also show that with the new convolution theorem it is convenient to implement in the designing of multiplicative filters through both the new convolution in the time domain and the product in the LCT domain.

1. Introduction

The convolution and product theorems are of importance in several application areas [1–4], including the designing of filters and the reconstruction of signals. It is well-known that the conventional convolution of two signals is equivalent to the product of the signals' Fourier transforms (FTs) [1,2]. This is the classical convolution and product theorem for the FT [1,2] which can be seen from equations reproduced as follows:

$$f(t)*g(t)=\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\mathrm{d}\tau, \quad (1)$$

$$f(t)*g(t)\overset{FT}{\longleftrightarrow}F(\omega)G(\omega), \quad (2)$$

where $*$ represents the conventional convolution operation, and $F(\omega)$ and $G(\omega)$ denote the FTs of the signals $f(t)$ and $g(t)$, respectively.

The linear canonical transform (LCT) is a three-free-parameter class linear integral transform [4–7] and includes the FT [1,2], the fractional Fourier transform (FRFT) [3,8–10], the Fresnel transform (FST) [11], the Lorentz transform [12], and the scaling operations as its special cases. The LCT can be considered as a quadratic phase system (QPS), which is one of the most important optical systems and is implemented with an arbitrary number of thin lenses and propagation through free space in the Fresnel approximation or through sections of graded-index media [3,24]. Then, it was applied to the analysis of the optical systems in the early years [5,6,13–15]. With in-depth research on the LCT, it has found many applications in some others fields, such as radar and sonar systems analysis, pattern recognition, time–frequency analysis, and image watermarking [3,4,16–21]. Particularly, the LCT is very useful and effective in non-stationary signal processing since it can be regarded as the decomposition of a signal based on a non-orthogonal basis [18–21]. Meanwhile, some fundamental theories and concepts associated with the LCT have been established, for example the convolution and product theorems [22–28], the uniform and nonuniform sampling theorems [29–34], and the uncertainty principles [35–39].

Recently, the convolution and product theorems for the LCT have attracted much attention in the literature, and then the convolution and product theorems in the FT [1,2] and FRFT [3,8–10] domains have been extended into the LCT domain in terms of different convolution operations [22–28]. However, none of them generalize very nicely and simply the classical result given by (1 and 2). Specifically, Deng et al. formulated a convolution and product theorem for the LCT [22] on the basis of the conventional convolution operation shown in (1). It takes the form

$$f(t)*g(t)\overset{LCT}{\longleftrightarrow}\frac{1}{|a|}e^{j\frac{c}{2a}u^2}\int_{-\infty}^{+\infty}F_A(v)e^{-j\frac{c}{2a}v^2}g\left(\frac{u-v}{a}\right)\mathrm{d}v, \quad (3)$$

where $F_A(u)$ denotes the LCT of $f(t)$ with parameter matrix $A=(a, b; c, d)$. Based on this convolution theorem, it is difficult to achieve multiplicative filters by means of the product in the LCT domain

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since the right-hand-side of (3) cannot be reduced to a simple multiplication as with that of (2). Due to this consideration, Deng et al. [22] and Wei et al. [23,24] proposed a new convolution theorem for the LCT pass utilizing a modified convolution operation. It is given by

$$f(t)\overset{A}{*}g(t) = \frac{1}{\sqrt{j2\pi b}}e^{-j\frac{a}{2b}t^2}((f(t)e^{j\frac{a}{2b}t^2}) * (g(t)e^{j\frac{a}{2b}t^2})) = \frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)e^{j\frac{a}{b}\tau(\tau-t)}\mathrm{d}\tau, \quad (4)$$

$$L_A[f(t)\overset{A}{*}g(t)](u) = F_A(u)G_A(u)e^{-j\frac{d}{2b}u^2}, \quad (5)$$

where $\overset{A}{*}$ denotes the modified convolution operation for the LCT. With the above two equations, multiplicative filters in the LCT domain can be achieved. Although this convolution theorem has the elegance and simplicity in both time and LCT domains, it does not exactly preserve the classical result for the FT as there contains an extra chirp multiplier in the right-hand-side of (5). Meanwhile, it must be emphasized that such a chirp multiplier may impose difficulty in real applications with respect to the impossibility of generating a chirp signal accurately in practical engineering. For this, due to a generalized translation and delay operator Wei et al. deduced a generalization convolution structure in the LCT domain [25,26] in order to eliminate the chirp multiplier. Here, we show it below:

$$f(t)\overset{A}{\Theta}g(t) = \frac{1}{2\pi|b|}\int_{-\infty}^{+\infty} f(\tau)e^{-j\frac{a}{2b}(t^2-\tau^2)} \times \left(\int_{-\infty}^{+\infty} G_A(u)e^{j\frac{1}{b}u(t-\tau)}\mathrm{d}u\right)\mathrm{d}\tau, \quad (6)$$

$$f(t)\overset{A}{\Theta}g(t)\overset{LCT}{\longleftrightarrow}F_A(u)G_A(u), \quad (7)$$

where $\overset{A}{\Theta}$ denotes the generalized convolution operation for the LCT. From (7), it is easier to implement in the designing of multiplicative filters through the product in the LCT domain than some suggested in [22–24]. As shown in [26], by using (6) to achieve the multiplicative filters, although it exhibits the same computational complexity as those introduced in [22–24], the generalized convolution expression given by the right-hand-side of (6) is a triple integral form which is complicated to be transformed into a single integral form as with that of (1). Moreover, Shi et al. presented a generalized convolution theorem associated with the LCT through defining a generalized canonical convolution operator [28], that is,

$$f(t)\Xi_{A_1,A_2,A_3}g(t) = \int_{-\infty}^{+\infty} f(\tau)(\mathrm{T}_\tau^{A_1}g)(t)\phi_{a_1,a_2,a_3}(t,\tau)\mathrm{d}\tau, \quad (8)$$

$$L_{A_3}[f(t)\Xi_{A_1,A_2,A_3}g(t)](u) = \epsilon_{d_1,d_2,d_3}(u)F_{A_1}\left(\frac{ub_1}{b_3}\right)F_{A_2}\left(\frac{ub_2}{b_3}\right), \quad (9)$$

where Ξ_{A_1,A_2,A_3} denotes the generalized canonical convolution operator. This convolution theorem includes the classical result given by (1 and 2), the convolution theorem for the FRFT introduced in [3,10], and the convolution theorem for the LCT shown in (4 and 5) as its special cases. Naturally, it has three different matrices for the LCT, i.e., nine free parameters, and hence it lacks elegance and simplicity comparable to the classical result.

In this paper, we obtain a new convolution structure for the LCT pass through a new convolution operation. Differing from those derived in [22–28], the new convolution theorem has the elegance and simplicity in both time and LCT domains comparable to the classical result for the FT. Certainly, it is easy to implement in the designing of multiplicative filters through both the new convolution in the time domain and the product in the LCT domain.

The remainder of this paper is structured as follows. Section 2 simply reviews the definition of LCT. In Section 3, a new kind of convolution structure for the LCT is proposed through using a new convolution operation. In Section 4, the new convolution theorem is presented to design multiplicative filters in the LCT domain from the view points of the new convolution in the time domain and the product in the LCT domain. Finally, Section 5 concludes this paper.

2. Linear canonical transform

The LCT is equivalent to a QPS, and therefore, it can be defined as the output light field of the QPS [3,24]

$$F_A(u) = L_A[f](u) = \begin{cases} \int_{-\infty}^{+\infty} f(t)K_A(u,t)\mathrm{d}t, & b \neq 0 \\ \sqrt{d}e^{j\frac{cd}{2}u^2}f(du), & b = 0 \end{cases}, \quad (10)$$

where $f(t)$ stands for the input light field, $F_A(u)$ represents the output light field, and the LCT kernel has the form

$$K_A(u,t) = \frac{1}{\sqrt{j2\pi b}}e^{j\left(\frac{d}{2b}u^2-\frac{1}{b}ut+\frac{a}{2b}t^2\right)}, \quad (11)$$

and where the parameter matrix $A=(a, b; c, d)$ and the parameters a, b, c, d are real numbers satisfying $ad-bc=1$.

From (10), the LCT is essentially a scaling and chirp multiplication operations when $b=0$, and then it is of no particular interest to our research. Therefore, we merely discuss the LCT in the case of $b \neq 0$ in this paper. The LCT has many important and useful properties [4], for instance the additivity, the reversibility, and the time, frequency and time–frequency shift properties. Since these properties will be used in the later, and then they are detailedly listed as follows:

The additivity of LCT [4]. Let $A=A_2A_1$, then

$$L_{A_2}[L_{A_1}[f]](u) = L_A[f](u). \quad (12)$$

The reversibility of LCT [4]. A signal $f(t)$ can be expressed by the LCT of $F_A(u)$ with parameter matrix $A^{-1}=(d, -b; -c, a)$, that is,

$$L_{A^{-1}}[L_A[f]](t) = f(t). \quad (13)$$

The time shift property of LCT [4]. The LCT of $\bar{f}(t)=f(t-\tau)$ can be presented by

$$L_A[\bar{f}](u) = e^{j(c\tau u-\frac{ac}{2}\tau^2)}L_A[f](u-a\tau). \quad (14)$$

The frequency shift property of LCT [4]. The LCT of $\widehat{f}(t)=f(t)e^{j\mu t}$ is given by

$$L_A[\widehat{f}](u) = e^{j\left(d\mu u-\frac{bd}{2}\mu^2\right)}L_A[f](u-b\mu). \quad (15)$$

The time–frequency shift property of LCT [4]. The LCT of $\widetilde{f}(t)=f(t-\tau)e^{j\mu t}$ takes the form

$$L_A[\widetilde{f}](u) = e^{j(c\tau+d\mu)u}e^{-j\left(\frac{ac}{2}\tau^2+bc\tau\mu+\frac{bd}{2}\mu^2\right)}L_A[f](u-a\tau-b\mu). \quad (16)$$

3. New convolution and product theorem for the LCT

In this section, a new convolution structure for the LCT is derived from defining a new convolution operation.