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Chaos, bifurcation, coexisting attractors and circuit design of a three-dimensional continuous autonomous system

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ABSTRACT

This letter investigates the complex dynamical behaviors of a three-dimensional continuous autonomous system which is described as $\dot{x} = ax - yz$, $\dot{y} = -by + xz$, $\dot{z} = -cz + x^2$. Some new results are presented by further research. The chaos and bifurcation of the system are analyzed. It proves that the system occurs double Hopf bifurcation at the equilibria. Also, study shows that the system coexist multiple attractors including point attractors, periodic attractors and chaotic attractors. Electronic circuit is also designed for realizing the chaos of the system.

1. Introduction

The research of the low-dimensional ordinary differential autonomous system is of great significance. It is widely accepted that the three-dimensional continuous system with quadratic nonlinearities usually has complex dynamical behaviors, including chaos, bifurcation, multistability, coexisting attractors, etc. The chaos is a very important phenomenon which is characterized by the highly sensitivity to the initial conditions. There are many examples of three-dimensional systems that perform chaotic behavior [1–8]. One notable example is the Lorenz system described by $\dot{x} = my - mx$, $\dot{y} = nx - y - xz$, $\dot{z} = xy - pz$, which has two nonlinear terms and five linear terms [9]. Since the discovery of the Lorenz system, the study of chaos has been in vogue. Bifurcation and chaos are always inseparable. The bifurcation is regarded as an essential factor to cause chaos. If a small smooth change of the system parameters yields a topological change in system behaviors, then we may say the system occurs a bifurcation. Three-dimensional continuous systems often appear different types of bifurcation, including fold bifurcation, flip bifurcation, pitchfork bifurcation and Hopf bifurcation [10–13]. Many researchers follow awfully with interest the analysis of chaos and bifurcation in recent years. The coexisting attractors is a dynamic behavior which is closely related to the initial condition of the system. For given system parameters, the state of the system changes along with the initial condition changes. The point attractors, periodic attractors, chaotic attractors and other types of attractors may simultaneously produce in a system. Some simple three-dimensional systems have been shown to have multiple attractors, see Refs. [14–17]. The study of the coexisting attractors has been gradually carried out by scholars.

This paper aims to reveal the complex dynamical behaviors of a three-dimensional continuous autonomous system presented in Refs. [18,19]. By further research, some new results of the system are presented. The chaotic attractor of the system is analyzed by numerical simulation and realized by electronic circuit. The Hopf bifurcation of the system is analyzed in detail. Double Hopf bifurcation are occurs in a pair of equilibria of the system as the parameter changes. Surprisedly, the coexisting attractors is found to exist in the system. Simulations intuitively show the coexisting attractors.

The paper is organized as follows. In Section 2, the model of the system and the stability of the equilibria are presented. In Section 3, the existence of Hopf bifurcation is proved. In Section 4, the dynamic evolution and the coexisting attractors are numerically given. In Section 5, the circuit implementation of the system is investigated. Finally, conclusions are stated in Section 6.

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2. System description

The mathematical model of the three-dimensional system proposed in Refs. [18,19] is described as follows:

$$\begin{cases} \dot{x} = ax - yz, \\ \dot{y} = -by + xz, \\ \dot{z} = -cz + x^2. \end{cases} \tag{1}$$

where x, y, z are state variables, a, b, $c \in R^+$ are constant parameters. The system (1) has three quadratic nonlinearities, and the nonlinearities are the main cause of chaos and bifurcation in the system (1). For $a = 6$, $b = 12$, $c = 4$, system (1) has a chaotic attractor as shown in Fig. 1. The corresponding Lyapunov exponents are calculated numerically as $LE_1 = 0.7286$, $LE_2 = 0$, $LE_3 = -10.7286$. Accordingly, the Lyapunov dimension of system (1) is fractional as given by $D_L = 2 - LE_1/LE_3 = 2.0679$. It is obvious that the attractor in system (1) is chaotic under parameters $a = 6$, $b = 12$, $c = 4$.

Let $\dot{x} = \dot{y} = \dot{z} = 0$, then three equilibria of the system (1) are obtained as

$$O(0, 0, 0),\quad Q_1(\sqrt[4]{abc^2}, \sqrt[4]{a^3c^2/b}, \sqrt{ab}),$$
$$Q_2(-\sqrt[4]{abc^2}, -\sqrt[4]{a^3c^2/b}, \sqrt{ab}).$$

The eigenvalues of Jacobian matrix at O are $\lambda_1 = a$, $\lambda_2 = -b$, $\lambda_3 = -c$. Since $\lambda_1 = a > 0$, then O is an unstable point. The Jacobian matrix of system (1) at $Q_i (i = 1, 2)$ is

$$J_1 = \begin{pmatrix} a & -z & -y \\ z & -b & x \\ 2x & 0 & -c \end{pmatrix},$$

The characteristic equation of J_1 is

$$\lambda^3 + (b + c - a)\lambda^2 + (a + b)c\lambda + 4abc = 0, \tag{2}$$

According to the Routh–Hurwitz criterion, $Q_i(i = 1, 2)$ is stable under the following conditions:

$$\begin{cases} b + c - a > 0, \\ b^2 - 4ab - a^2 + (a + b)c > 0. \end{cases}$$

In [18], the authors investigated some basic dynamic behaviors of system (1) by theoretical analysis and numerical simulation only for a given system parameters. In [19], the authors applied the simulation method including phase portraits, bifurcation diagram, Lyapunov exponent spectrum, Poincaŕe map for considering the dynamics of the system (1). The nonlinear amplitude adjuster, phase reversal and modulation factor are also numerically analyzed. In this presented paper, we will give a detailed and deeper description of the dynamic behaviors of the system (1). The chaotic behavior, Hopf bifurcation, coexisting attractors are considered. Electronic circuit on the Multisim software is designed for realizing the chaotic attractor of the system (1).

3. Hopf bifurcation

Here we will investigate the Hopf bifurcation at equilibrium $Q_i(i = 1, 2)$. Assume that the Eq. (2) has a pure imaginary root $\lambda = \sigma i$, $\sigma > 0$, then

$$(\sigma i)^3 + (b + c - a)(\sigma i)^2 + (a + b)c\sigma i + 4abc = 0.$$

Separating the real part and the imaginary part of the above equation, one has

$$\begin{cases} (a + b)c - \sigma^2 = 0, \\ 4abc - (b + c - a)\sigma^2 = 0, \end{cases} \tag{3}$$

and

$$\begin{cases} c_0 = \dfrac{a^2 + 4ab - b^2}{a + b}, \\ \sigma_0 = \sqrt{a^2 + 4ab - b^2}, \end{cases} \tag{4}$$

So the Eq. (2) exists a root $\lambda = \sigma i$, $\sigma > 0$ as long as $a^2 + 4ab - b^2 > 0$. Differentiating Eq. (2) with respect to c, then

$$\frac{d\lambda}{dc} = -\frac{\lambda^2 + (a + b)\lambda + 4ab}{3\lambda^2 + 2(b + c - a)\lambda + (a + b)c},$$

and

$$\mathrm{Re}\left(\frac{d\lambda}{dc}\right)|_{c=c_0,\lambda=\sigma_0 i} = -\frac{(a + b)(a^2 + 4ab - b^2)}{2(a + b)(a^2 + 4ab - b^2) + 8ab} < 0. \tag{5}$$

Hence, by the Hopf bifurcation theory proposed in Refs. [20,21], system (1) undergoes a Hopf bifurcation at Q_i when $c = c_0$. By computing the first Lyapunov coefficient defined by Kuznetsov [21], we can give more detailed description of the Hopf bifurcation of system (2). For simplicity, we assume $b = 3a$. So we just need to analyze the Hopf bifurcation on the line $c = a$, $b = 3a$ except the origin point.

Let C^n is a linear space defined on the complex number field C. For $X = (x_1, x_2, \ldots, x_n)^T$, $Y = (y_1, y_2, \ldots, y_n)^T$ with $x_i, y_i \in C(i = 1, 2, \ldots, n)$, $\langle X, Y\rangle = \sum_{i=1}^{n} \bar{x}_i y_i$ is defined as the inner product of X, Y. Consider the nonlinear system as follows:

$$\dot{X} = AX + F(X), \tag{6}$$

where $F(X)$ is described as

$$F(X) = \frac{1}{2}S(X, X) + \frac{1}{6}M(X, X, X) + o(\|x\|^4), \tag{7}$$

where $\|X\| = \sqrt{\langle X, X\rangle}$, $S(X, X)$ and $M(X, X, X)$ are bilinear and trilinear functions. Suppose $\lambda_{1,2} = \pm\omega i$, $\omega > 0$ are the only pair of pure imaginary eigenvalues of A, v is a eigenvector respect to eigenvalue λ_1, then $Av = i\omega v$, $A\bar{v} = -i\omega\bar{v}$. Let u is a adjoint eigenvector with $A^T u = -i\omega u$, $A^T\bar{u} = i\omega\bar{u}$, $\langle u, v\rangle = 1$. According to the Ref. [21], the first Lyapunov coefficient is given by

$$f_0 = \frac{1}{2\omega}\mathrm{Re}\left[\langle u, M(v, v, \bar{v})\rangle - 2\langle u, S(v, A^{-1}S(v, \bar{v}))\rangle \right.$$
$$\left. + \langle u, S(\bar{v}, (2i\omega I - A)^{-1}S(v, v))\rangle\right]. \tag{8}$$

The Jacobian matrix at Q_1 is rewritten as

$$A = \begin{pmatrix} a & -\sqrt{3}a & -\sqrt[4]{1/3}a \\ \sqrt{3}a & -3a & \sqrt[4]{3}a \\ \sqrt[4]{48}a & 0 & -a \end{pmatrix}, \tag{9}$$

The eigenvalues of A are $\lambda_{1,2} = \pm 2ai$, $\lambda_3 = -3a$. The corresponding vectors u, v with $Av = i2av$, $A^T u = -i2au$ of A are given as

$$v = \begin{pmatrix} \dfrac{2i + 1}{2\sqrt[4]{3}} \\ \dfrac{\sqrt[4]{3}}{2} \\ 1 \end{pmatrix},\qquad u = \begin{pmatrix} \dfrac{\sqrt[4]{3}i}{2} \\ \dfrac{\sqrt{3}\sqrt[4]{3}(2 - 3i)}{26} \\ \dfrac{5 - i}{13} \end{pmatrix}.$$

From system (1), the bilinear and trilinear functions are

$$S(Y, Y') = \begin{pmatrix} -yz' \\ xz' \\ xx' \end{pmatrix},\quad M(Y, Y', Y'') = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $Y = (x, y, z)^T$, $Y' = (x', y', z')^T$, $Y'' = (x'', y'', z'')^T$.