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A semi-analytical solution of foam drainage equation by Haar wavelets method

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ABSTRACT

In this paper, Haar wavelets method (HWM) is applied to compute the numerical solutions of the foam drainage equation. The mathematical base of the method is presented. First, the time derivative is discretized by a forward difference scheme and then a quasilinearization technique is used to linearize the foam drainage equation. Finally, we solve a system of linear equations which is obtained by applying the Haar wavelet method for discretizing the space derivatives. Obtained results by HWM are very similar to the exact solutions. Further, a comparison between our results and results which are obtained by HPM, HPTM, LDM and ADM is presented. Numerical results show that our method works better than the other methods.

1. Introduction

One of the interesting and applicable partial differential equations is the foam drainage problem. The governing equation of this problem is expressed as follows

$$\frac{\partial \Psi}{\partial t} + \frac{\partial}{\partial x}\left(\Psi^2 - \frac{\sqrt{\Psi}}{2}\frac{\partial \Psi}{\partial x}\right) = 0, \quad (1)$$

where Ψ is the cross section of a channel formed where three films meet which usually indicated as *Platean border*, x and t are scaled position and time coordinates, respectively. A complex fluid or a liquid foam is an example of a soft matter with a well-defined structure. Joseph Platean, for the first time, clearly described such structures in the 19th century.

Foams are very well known for both common people and scientists, because of their everyday utilization [1,2]. They are common in food and personal care products such as creams, lotions, etc and foams are often are used during scrubbing and clothes cleaning [3]. One may find many applications of foams in mineral processing, fire fighting, food, chemical industries and structural material sciences in Ref. [4]. Anybody has everyday direct contact with foams. Washing dishes, shampooing hair, eating chocolate bars and chocolate mouses desserts are only a few examples. From above applications one can see the great importance of foams in many technological processes and their properties which have intensive studies from scientific and practical point of view. This is why foams have been of great interest for academic researches. Some popular mentioned applications include the use of foams for reducing the impact of explosions and for cleaning up oil spins. There are now many applications of polymetric foam [5] and more recently metallic foams which are foams made of metals such as aluminum [6]. Polymetric foams are used in cushions, structural materials and packing [5]. Ceramic, glass and metal foams can also be made, for more details see e.g. [7]. Uniformity of the foam is important for designers who interested in these applications. Gravitational drainage of the liquid is one mechanism leading to nonuniformity. Resent researches on foams have centered on three topics which are often treated separately, but are, interdependent. These topics are drainage, coarsening and rheology. Here, we concentrate on a quantitative description of the coupling of drainage. The flow of liquid through the liquid-filled channels, which are called Plateau borders, and intersections of four channels between the bubbles, driven by gravity and capillarity is called foam drainage. Foam drainage has a very important role in foam stability. For example in drying process of a foam, its structure becomes fragile [8]. In spite of many applications and numerous scientific investigations of properties and mechanics of foams, dynamics of foam drainage has only recently been examined in detail [9–11]. Also there are another researches to handle the foam drainage equation. Helal and Mehanna [12] used the Adomian decomposition method (ADM) and tanh method to handle the foam drainage equation. Darvishi and Khani [13] presented a series solution of the foam drainage equation by homotopy analysis method. Khani et al. [14] used the exp-function method to obtain some new solitary wave and periodic solution for the

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problem. Mirmoradi et al. [15] applied the variational iteration method (VIM) to solve the foam drainage equation.

Fereidoon et al. [16] applied homotopy perturbation method (HPM) to obtain semi-analytical solutions for the foam drainage equation. Darvishi et al. [17] obtained some traveling wave solutions for the foam drainage equation by modified *F*-expansion method. Khan [18] applied the Homotopy perturbation transform method (HPTM) for solving the foam drainage equation. Khan and Gondal [19] using Laplace decomposition method (LDM) obtained numerical solutions for the foam drainage equation.

In recent years the wavelets have been used for the solution of partial differential equations (PDEs). Different types of wavelets and approximating functions have been used in numerical solution of differential equations. The Haar wavelets method (HWM) is the most simple one among the different wavelet families which are defined by an analytical expression. Due to its simplicity, HWM is very effective to solve ordinary and partial differential equations. The notion of wavelets is introduced by Alfred Haar in 1910. Expression of Haar wavelets is very simple. Besides they have orthogonal and normalization with compact support properties. Therefore, the Haar wavelets are very efficient and effective tools to solve the nonlinear systems in physics, biology, chemical reactions and fluid mechanics, for more details, see [20–25].

In this paper, we use the HWM to obtain a semi-analytical solution for the foam drainage equation. In this composite scheme, first of all, we discretize time derivative terms by a forward difference scheme and linearize the nonlinear terms using a quasilinearization technique [26] to reduce the original equation into a system of ordinary differential equations. Then we apply the Haar wavelets method which leads to a system of algebraic equations. To solve the system of algebraic equations we use the **solve** package in Maple 16.00.

This paper is organized as follows: In the next section we present a brief introduction on preliminaries of Haar wavelets and its integrals. Section 3 describes the quasilinearization technique for nonlinear terms. Convergence of method is discussed in Section 4. In Section 5, Haar wavelets method is used for solving the foam drainage equation. Numerical results are presented in Section 6 which Eq. (1) with two different initial conditions is solved. Finally, the paper in concluded in Section 7.

2. Haar wavelets

The Haar wavelets family $\{h_i(x)\}$ is defined as a group of orthogonal square waves with magnitude ± 1 in some intervals and zero elsewhere as follows

$$h_i(x) = \begin{cases} 1 & x \in [\xi_1, \xi_2), \\ -1 & x \in [\xi_2, \xi_3), \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where the notations $\xi_1 = \frac{k}{m}$, $\xi_2 = \frac{k+0.5}{m}$, $\xi_3 = \frac{k+1}{m}$ are introduced. The integer $m = 2^j$, $(j = 0, 1, \ldots, J)$ indicates the level of the wavelets, where J is the maximal level of resolution and $k = 0, 1, \ldots, m-1$ is the translation parameter. The subscript i can be expressed as $i = m + k + 1$, such that in the case of $m = 1$, $k = 0$ we have $i = 2$; the maximal value of i is $i = 2M = 2^{J+1}$. For $i = 1$, the function $h_1(x)$ is a scaling function for the family of the Haar wavelets as

$$h_1(x) = \begin{cases} 1 & x \in [0, 1), \\ 0 & \text{otherwise}. \end{cases}$$

We introduce the following notations; for $i > 1$

$$P_{i,1}(x) = \int_0^{x_l} h_i(x)\,dx, \tag{3}$$

$$P_{i,n+1}(x) = \int_0^{x} P_{i,n}(x)\,dx, \quad n = 1, 2, \ldots \tag{4}$$

where for $l = 1, 2, \ldots, 2M, x_l = \frac{l-0.5}{2M}$ are the collocation points. By using Eqs. (2) and (4) we have

$$P_{i,n}(x) = \begin{cases} 0 & x < \xi_1, \\ \dfrac{(x-\xi_1)^n}{n!} & x \in [\xi_1, \xi_2), \\ \dfrac{(x-\xi_1)^n}{n!} - 2\dfrac{(x-\xi_2)^n}{n!} & x \in [\xi_2, \xi_3), \\ \dfrac{(x-\xi_1)^n}{n!} - 2\dfrac{(x-\xi_2)^n}{n!} + \dfrac{(x-\xi_3)^n}{n!} & x > \xi_3. \end{cases} \tag{5}$$

It is well-known that any integrable function $u(x) \in L^2[0, 1)$ can be expanded by a Haar series with an infinite number of terms as follows

$$u(x) = \sum_{i=1}^{\infty} a_i h_i(x). \tag{6}$$

The above series terminates at finite terms if $u(x)$ is a piecewise constant or can be approximated as a piecewise constant function during each subinterval, then $u(x)$ will be terminated at finite terms, i.e.,

$$u(x) = \sum_{i=1}^{2M} a_i h_i(x) = a_{(2M)}^T h_{(2M)}(x), \tag{7}$$

where the coefficients $a_{(2M)}^T$ and the Haar function vector $h_{(2M)}(x)$ are defined as

$$a_{(2M)}^T = [a_1, a_2, \ldots, a_{2M}]$$

$$h_{(2M)}(x) = [h_1(x), h_2(x), \ldots, h_{2M}(x)]^T,$$

where superscript T shows the transpose operator and $M = 2^J$.

3. Quasilinearization

The quasilinearization approach [26] is a generalized Newton–Raphson technique for functional equations. It converges quadratically to the exact solution. Also, if there is a convergence at all, it has a monotonic convergence.

Consider the nonlinear nth order differential equation

$$L^n u(x) = f(u(x), u'(x), \ldots, u^{n-1}(x), x). \tag{8}$$

Application of quasilinearization technique to (8) yields

$$L^n u_{r+1}(x) = f(u_r(x), u_r'(x), \ldots, u_r^{n-1}(x), x) + \sum_{j=0}^{n-1} (u_{r+1}^j(x) - u_r^j(x)) f_{u^j}(u_r(x), u_r'(x), \ldots, u_r^{n-1}(x), x), \tag{9}$$

with the initial and boundary conditions at $(r+1)$th iteration, where n is the order of the differential equation. Eq. (9) is always a linear differential equation and can be solved recursively, where $u_r(x)$ is known and one can use it to get $u_{r+1}(x)$.