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# Exponential synchronization of chaotic system and application in secure communication

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ABSTRACT

This paper is concerned with the exponential synchronization of the chaotic system without linear term and its application in secure communication. The synchronization analysis is carried out by exponential stability theorem. Also secure communication is obtained by masking method and parameter modulation between transmitter and receiver. The error system exponentially is converged to zero. Numerical simulations are presented to illustrate the ability and effectiveness of proposed method.



## 1. Introduction

Chaos, which is an interesting phenomenon in nonlinear dynamical systems, has been studied over the last four decades [12,15,19,20,24,25]. Chaotic and hyperchaotic systems are nonlinear deterministic systems that display complex and unpredictable behavior. Also these systems are sensitive respect to initial conditions. The chaotic and hyperchaotic systems have many important applications in nonlinear sciences, such as laser physics, secure communications, nonlinear circuits, control, neural networks, chemical reactor and active wave propagation [4,10,12,13,17,18,21,22].

The synchronization of chaotic systems has been investigated since its introduction in the paper by Pecora and Carrol in 1990 [20] and has been widely investigated in many fields, such as physics, chemistry, ecological sciences and secure communications [2,11,24]. Various techniques and methods have been proposed to achieve chaos synchronization, such as adaptive control, impulsive control, active control and nonlinear control and exponential method [1,5,6,23]. Fortunately, some existing methods of synchronizing can be generalized to anti-synchronization of chaotic systems [1,3,16].

Most of mentioned approaches to achieve the chaotic synchronization are based on the asymptotical stability that is provided by Lyapunov stability theorem. Exponential stability [14] is a stronger type of stability which is compared with asymptotical stability. Liao and Yu [6], Yan and Yu [7] studied the exponential synchronization of the family Rössler dynamical systems. In 2009, Sun [9] used the exponential synchronization between two classes of chaotic systems. Yang in 2013 used it for exponential synchronization of a four-dimensional chaotic system [5].

Since 1992, secure communication based on synchronization of chaotic dynamical systems has been developed. The general idea for transmitting information via chaotic systems is that, an information signal is embedded in the transmitter system which produces a chaotic signal. The sending information signal, is recovered by the receiver system.

The techniques of chaotic communication include chaos masking, chaos modulation and chaos shift keying. The idea of chaos masking is that the information signal is added directly to the transmitter. Chaos modulation is based on the master-slave synchronization, where the information signal is injected into the transmitter as a nonlinear filter. Chaos shift keying is supposed the information signal to be binary, and it is mapped into the transmitter and the receiver. In these three cases, the information signal can be recovered by a receiver, if the transmitter and the receiver were synchronized [26,30,31].

In 1993, Cuomo et al. [27] developed the additive chaos masking approach. Dedieu et al. [28] presented the chaotic shift keying or the chaotic switching approach in 1993. Then in 1996, Yang and Chua [32] introduced the chaotic parameter modulation method, where the information signal is used to modulate the parameters of the chaotic system in the transmitter. Yang et al. [29] introduced a novel secure communication scheme. In this scheme, the information signal is encrypted by an encryption rule with a key generated from the chaotic system in the transmitter.

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Recently, several new chaotic systems were introduced. The synchronization of these systems via novel schemes was discussed, then this systems with new synchronization methods were used for secure communication based on chaotic systems [33–36].

A new chaotic system without linear term and its impulsive synchronization was introduced in [8]. This paper investigates the exponential synchronization of the chaotic system without linear term and its application in secure communication. Synchronization of this system is based on the exponential stability theorem.

The rest of the paper is organized as follow. Section 2 briefly introduces the chaotic system without linear term and describes the exponential synchronization. Section 3, investigates the secure communication via masking and modulation method, based on exponential synchronization. Concluding remarks are given in Section 4.

## 2. Exponential synchronization of chaotic system without linear term

In 2014, Y. Xu and Y. Wang [8] introduced a new chaotic system without linear term, as follows:

$$\begin{cases} \dot{x_1} = \ln(a + e^{x_2 - x_1}) \\ \dot{x_2} = x_1 x_3 \\ \dot{x_3} = b - x_1 x_2, \end{cases} \tag{2.1}$$

where $x_1$, $x_2$ and $x_3$ are the state variables and $a$ and $b$ are real parameters. By choosing $a = 0.1$, $b = 0.25$ the system (2.1) is chaotic. For further information about the (2.1) see [8].

Suppose (2.1) is the master system and define slave system, as follow:

$$\begin{cases} \dot{y_1} = \ln(a + e^{y_2 - y_1}) + u_1 \\ \dot{y_2} = y_1 y_3 + u_2 \\ \dot{y_3} = b - y_1 y_2 + u_3, \end{cases} \tag{2.2}$$

where $u_i (1 = 1, 2, 3)$ are control functions.

Let $e = y - x$ is the error vector of states, then the error system is:

$$\begin{cases} \dot{e_1} = \ln\left(\dfrac{a + e^{e_2 - e_1} e^{x_2 - x_1}}{a + e^{x_2 - x_1}}\right) + u_1 \\ \dot{e_2} = e_1 e_3 + e_1 x_3 + e_3 x_1 + u_2 \\ \dot{e_3} = -(e_1 e_2 + e_1 x_2 + e_2 x_1) + u_3. \end{cases} \tag{2.3}$$

In what follows, the exponential synchronization and requirement Lemmas are presented.

**Definition 2.1.** [5]. The slave system (2.2) exponentially synchronizes with the master system (2.1) for any initial condition, if the solution of the error system (2.3) has the following estimation

$$E(t)E^T(t) \le A \exp(-\sigma(t - t_0)),$$

where $E(t) = [e_1(t), e_2(t), e_3(t)]$, $A$ is a positive constant depending on the initial value $E(t_0)$, and $\sigma$ is positive constant independent of $E(t_0)$ and is named the exponential convergence rate. Then, the zero solution of system (2.3) is exponentially stable and systems (2.2) and (2.1) are called exponential synchronization.

**Lemma 2.2.** [14,5]. *For the error system (2.3), if there is an existed positive definite quadratic polynomial $V(E(t)) \equiv V(t)$ such that*

$$\omega_1 E^T(t)E(t) \le V(t) \le \omega_2 E(t)E(t)^T \tag{2.4}$$

$$\dot{V}(t) \le -\omega_3 E(t)E(t)^T, \tag{2.5}$$

*where $\omega_1, \omega_2, \omega_3$ are positive constants and $\omega_1 \le \omega_2$, then the zero solution of system (2.3) is exponentially stable and systems (2.2) and (2.2) are called exponential synchronization.*

**Lemma 2.3.** [5]. *For any $\rho \in \mathbb{R}^+$;$X, Y \in \mathbb{R}$, the inequality $2|X||Y| \le \rho X^2 + \rho^{-1} Y^2$ holds.*

**Theorem 2.4.** *The master system (2.1) and the slave system (2.2) are exponential synchronized with feedback controls*

$$\begin{cases} u_1 = -(M + 2k_1 e_1) \\ u_2 = -k_2 e_2 \\ u_3 = -k_3 e_3, \end{cases} \tag{2.6}$$

*where constant $M$ and the feedback gains $k_i > 0 (i = 1, 2, 3)$ satisfies*

$$\begin{cases} M > \ln\left(\dfrac{a + e^{e_2 - e_1} e^{x_2 - x_1}}{a + e^{x_2 - x_1}}\right) \\ k_1 > \dfrac{1}{2}\left(\dfrac{M_3}{\rho_1} + \dfrac{M_2}{\rho_2}\right) \\ k_2 > \dfrac{1}{2} M_3 \rho_1 \\ k_3 > \dfrac{1}{2} M_2 \rho_2, \end{cases} \tag{2.7}$$

*where $\rho_i (i = 1, 2)$ are positive constant and $|x_i| < M_i (i = 1, 2, 3)$.*

**Proof.** *Let the quadratic function candidate be define as*

$$V(t) = \frac{1}{2}\left(\frac{1}{2} e_1^2 + e_2^2 + e_3^2\right) = EPE^T \tag{2.8}$$

*where the matrix $P = diag\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right)$ is diagonal and positive definite. Taking the time derivative of (3.16) along the error of state system, the control law (2.6), it yields*

$$\begin{aligned} \dot{V}(t) &= \frac{1}{2} e_1 \dot{e_1} + e_2 \dot{e_2} + e_3 \dot{e_3} \\ &= \frac{1}{2} e_1 \left(\ln\left(\frac{a + e^{e_2 - e_1} e^{x_2 - x_1}}{a + e^{x_2 - x_1}}\right) + u_1\right) + e_2(e_1 e_3 + e_1 x_3 + e_3 x_1 + u_2) \\ &\quad + e_3(-(e_1 e_2 + e_1 x_2 + e_2 x_1) + u_3). \end{aligned} \tag{2.9}$$

*Assume $\ln\left(\frac{a + e^{e_2 - e_1} e^{x_2 - x_1}}{a + e^{x_2 - x_1}}\right) < M$, $|x_i| < M_i (i = 1, 2, 3)$ and apply (2.6) in (2.9), we have:*

$$\begin{aligned} \dot{V}(t) &< -(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2) + x_3 e_2 e_1 - x_2 e_3 e_1 \\ &< -(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2) + M_3 |e_2||e_1| + M_2 |e_3||e_1|. \end{aligned} \tag{2.10}$$

*From lemma 2.3, for any $\rho_1, \rho_2 > 0$, can be got*

$$\begin{aligned} \dot{V}(t) &< -(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2) + \frac{M_3}{2}(\rho_1 e_2^2 + \rho_1^{-1} e_1^2) + \frac{M_2}{2}(\rho_2 e_3^2 + \rho_2^{-1} e_1^2) \\ &= -e_1^2\left(k_1 - \frac{M_3}{2\rho_1} - \frac{M_2}{2\rho_2}\right) - e_2^2\left(k_2 - \frac{M_3 \rho_1}{2}\right) - e_3^2\left(k_3 - \frac{M_2 \rho_2}{2}\right). \end{aligned} \tag{2.11}$$

*By chosen $K_i > 0$ as follow*

$$\begin{cases} k_1 > \dfrac{M_3}{2\rho_1} + \dfrac{M_2}{2\rho_2} \\ k_2 > \dfrac{M_3 \rho_1}{2} \\ k_3 > \dfrac{M_2 \rho_2}{2}, \end{cases} \tag{2.12}$$

*we have*

$$\dot{V}(t) < -\alpha E E^T, \tag{2.13}$$

*where $\alpha = \min\{k_1 - \frac{M_3}{2\rho_1} - \frac{M_2}{2\rho_2}, k_2 - \frac{M_3 \rho_1}{2}, k_3 - \frac{M_2 \rho_2}{2}\}$.*

*According to Lemma 2.2, the inequality (2.13) implies that system (2.2) exponentially synchronizes with system (2.1).* □

**Example 2.5.** To demonstrate and verify the validity of the proposed scheme, we present and discuss the numerical results for exponential synchronization. For synchronization systems (2.1) and (2.2) with controllers (2.6) are solved numerically by Matlab.