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A generalized convolution theorem for the special affine Fourier transform and its application to filtering

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ABSTRACT

The special affine Fourier transform (SAFT), which is a time-shifted and frequency-modulated version of the linear canonical transform (LCT), has been shown to be a powerful tool for signal processing and optics. Many properties for this transform are already known, but an extension of convolution theorem of Fourier transform (FT) is still not having a widely accepted closed form expression. The purpose of this paper is to introduce a new convolution structure for the SAFT that preserves the convolution theorem for the FT, which states that the FT of the convolution of two functions is the product of their Fourier transforms. Moreover, some of well-known results about the convolution theorem in FT domain, fractional Fourier transform (FRFT) domain, LCT domain are shown to be special cases of our achieved results. Last, as an application, utilizing the new convolution theorem, we investigate the multiplicative filter in the SAFT domain. The new convolution structure is easy to implement in the designing of filters.

1. Introduction

The special affine Fourier transform (SAFT) [1–4], also called the offset linear canonical transform (OLCT) [4] or the inhomogeneous canonical transform [3], is a six-parameter $(a, b, c, d, u_0, \omega_0)$ class of linear integral transform. The SAFT encompasses a number of important transform in digital signal processing and optical system modeling. The well-known signal processing operations, such as the Fourier transform (FT), the offset FT [3,4], the fractional Fourier transform (FRFT) [5,6], the offset FRFT [3,4], the Fresnel transform [7], the linear canonical transform (LCT) [8–10] and the scaling operations are all special case of the SAFT. With the progression of LCT theory, SAFT also has evolved as an interesting tool. The SAFT is more general and flexible than the original LCT for its two extra parameter u_0 and ω_0. Recently, along with applications of the LCT in the signal processing community [5,9,11–15], the role of the SAFT for signal processing has also been noticed. It has found many applications in optics, signal processing, and many other applications [3,4,16–18].

With intensive research of the SAFT, many properties have been found including time shift, phase shift, scaling, differentiation, integration and so on [3,16]. Simultaneously, as the generalization of FT, the relevant theory of SAFT has been developed including the convolution theorem, uncertainty principle, sampling theory and so on [16–18], which are generalizations of the corresponding properties of the FT, FRFT and LCT [3,9,19–28]. Conventional convolution operations for FT are fundamental in the theory of linear time-invariant (LTI) system [9]. The output of any continuous-time LTI system is found via the convolution of the input signal with the system impulse response. As the SAFT has found wide applications in optic and signal processing fields, it is theoretically interesting and practically useful to consider the convolution theory in the SAFT domain. However, the convolution theorems don't have the elegance and simplicity comparable to that of the FT, which states that the FT of the convolution of two functions is the product of their Fourier transforms.

Convolution theorem for a linear integral transform can be formulated in several ways. Recently, Xiang and Qin [16] introduced a new convolution operation that is more suitable for the SAFT and by which the SAFT of the convolution of the two functions is the product of their SAFTs and a phase factor. However, on the one hand the convolution theorem for the SAFT derived in [16] with the modified convolution operation also contains an extra chirp factor and hence does not exactly parallel the theorem given by FT. On the other hand, there possesses different chirp multiplications, which are difficult to implement in the engineering based on it is nearly impossible to generate a chirp signal accurately. In this paper, we propose a new convolution structure for the SAFT, which is different from the convolution structure derived in [16].

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Table 1
Some of the specific cases of the saft.

Parameter A	Corresponding transform
$A=(a, b, c, d, 0, 0)$	Linear canonical transform (LCT)
$A=(\cos\theta, \sin\theta, -\sin\theta, \cos\theta, u_0, \omega_0)$	Offset fractional Fourier transform (OFRFT)
$A=(\cos\theta, \sin\theta, -\sin\theta, \cos\theta, 0, 0)$	Fractional Fourier transform (FRFT)
$A=(0, 1, -1, 0, u_0, \omega_0)$	Offset Fourier transform (OFT)
$A=(0, 1, -1, 0, 0, 0)$	Fourier transform (FT)
$A=(1, b, 0, 1, 0, 0)$	Fresnel transform
$A=(1, 0, 0, 1, u_0, 0)$	Time shift
$A=(1, 0, 0, 1, 0, \omega_0)$	Frequency shift
$A=(d^{-1}, 0, 0, d, 0, 0)$	Time scaling

Based on the expression for the generalized translation in the SAFT domain, the generalized convolution theorem can be derived in the SAFT domain, which shows that the generalized convolution of two signals in time domain is equivalent to simple multiplication of their SAFTs in the SAFT domains. This result is an extension of the convolution theorem from the FT to the SAFT domain, and can be more useful in practical analog filtering in the SAFT domain. We also show that the convolution theorem in FT or FRFT domain can be looked as special cases of our achieved results.

The rest of the paper is organized as follows. In Section 2, we provide a brief review of the SAFT and convolution theory. A new convolution theorem for SAFT is derived based on generalized translation in Section 3. In Section 4, the multiplicative filter is investigated in the SAFT domain. The paper is concluded in Section 5.

2. Preliminaries

2.1. Special affine Fourier transform (SAFT)

The SAFT was introduced by Abe and Sheridan [1,2] who studied a transformation in phase space that was associated with a general, inhomogeneous, lossless linear mapping. The SAFT of a signal $f(t)$ with real parameter $A=(a, b, c, d, u_0, \omega_0)$ is defined as [1–4]

$$F_A(u) = S^A[f(t)](u) = \begin{cases} \int_{-\infty}^{+\infty} f(t) h_A(u,t)\,\mathrm{d}t, & b \neq 0 \\ \sqrt{\mathrm{d}}e^{j(cd/2)(u-u_0)^2 + j\omega_0 u} f[\mathrm{d}(u-u_0)], & b = 0 \end{cases} \tag{1}$$

where

$$h_A(u,t) = \sqrt{\frac{1}{j2\pi b}} e^{j(1/2b)\left[\mathrm{d}(u_0^2+u^2) - 2u(\mathrm{d}u_0 - b\omega_0) + 2t(u_0-u) + at^2\right]} \tag{2}$$

$a, b, c, d, u_0, \omega_0$ are real numbers satisfying $ad-bc=1$. We only consider the case of $b \neq 0$, since the SAFT is just a chirp multiplication operation if $b=0$. And without loss of generality, we assume $b>0$ in the following sections. The inverse of the SAFT is given by parameters $A^{-1}=(d, -b, -c, a, b\omega_0 - du_0, cu_0 - a\omega_0)$ as follows:

$$f(t) = S^{A^{-1}}[F_A(u)](t) = C \int_{-\infty}^{+\infty} F_A(u) h_{A^{-1}}(u,t)\,\mathrm{d}u \tag{3}$$

where $C = e^{j(1/2)\left(cdu_0^2 - 2adu_0\omega_0 + ab\omega_0^2\right)}$. This can be verified by using the Definition (1). The SAFT can model a number of optical operations such as rotation and magnification (see Table 1).

The SAFT has the following important space shift and phase shift properties [4,16], which are used to derive the new convolution theorems for SAFT in this paper.

Property 1. The space shift property

$$S^A[f(t-\tau)](u) = F_A(u - a\tau) e^{-j(ac\tau^2/2) + jc\tau(u-u_0) + ja\tau\omega_0} \tag{4}$$

Property 2. The phase shift property

$$S^A\left[f(t) e^{j\upsilon t}\right](u) = F_A(u - b\upsilon) e^{-j(bd\upsilon^2/2) + jd\upsilon(u-u_0) + jb\upsilon\omega_0} \tag{5}$$

Property 3. The space shift and phase shift properties

$$S^A\left[f(t-\tau)e^{j\upsilon t}\right](u) = F_A(u - a\tau - b\upsilon) e^{-j(ac\tau^2 + bd\upsilon^2/2) + j(c\tau + d\upsilon)(u-u_0) - jbc\tau\upsilon + j(a\tau + b\upsilon)\omega_0} \tag{6}$$

2.2. The convolution theory

Convolution and correlation operations are fundamental in the theory of LTI system. Moreover, convolution and correlation are widely used in signal processing, as well as in optics, in pattern recognition or in the description of image formation with incoherent illumination [16,24–30]. The convolution operation in FT domain is defined as

$$f(t) \otimes g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau)\,\mathrm{d}\tau \tag{7}$$

$$f(t) \otimes g(t) \xleftarrow{\mathrm{FT}} F(u) G(u) \tag{8}$$

where '$\otimes$' denotes the conventional convolution operation.

Recently, the convolution theorem has been derived in the SAFT domain by Xiang [16] as follows:

$$z(t) = \sqrt{\frac{1}{j2\pi b}} e^{j(d/2b)u_0^2} \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) e^{-j(a\tau/b)(t-\tau)}\mathrm{d}t \tag{9}$$

$$S^A[z(t)](u) = F_A(u) G_A(u) e^{(j/2b)[-du^2 + 2u(du_0 - b\omega_0)]} \tag{10}$$

The SAFT of the convolution of the two functions is the product of their SAFTs and a phase factor. However, the convolution theorem doesn't have the elegance and simplicity comparable to that of the FT.

3. New convolution structure for the SAFT based on generalized translation

In this section, we seek to modify the ordinary convolution structure using the generalized translation based on the kernel of the SAFT. We will derive the new convolution theorem for the SAFT based on the generalized translation. The new convolution theorem has the elegance and simplicity comparable to the classical result for the FT. Some of the well-known results about the convolution theorem in FT domain, FRFT domain, LCT domain are shown to be special cases of our achieved results.

3.1. Generalized translation and general framework of convolution theory

Generalized translation is necessary in dealing with the signal transforms with non-exponential kernels to have the translation or shift property of comparable simplicity to that of the Laplace or Fourier transforms [31].

If we consider a general signal transform and its Fourier-type inverse given by

$$f(t) = \int \rho(\omega) F(\omega) h(\omega, t)\,\mathrm{d}\omega \tag{11}$$