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Two approximation algorithms of error spectrum for estimation performance evaluation

Wei-Shi Peng [a,b,*], Yang-Wang Fang [a], Ren-Jun Zhan [b], You-Li Wu [a]

[a] School of Aeronautics and Astronautics Engineering, Air Force Engineering University, Xi'an 710038, Shaanxi, China
[b] School of Equipment Engineering, Armed Police Force Engineering University, Xi'an 710086, Shaanxi, China

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ABSTRACT

Error spectrum is a comprehensive metric for evaluation of estimation performance in that it is an aggregation of many incomprehensive measures. However, error spectrum requires computing the expectation of the *r*th power of the estimation-error-norm as using it to evaluate an estimator's performance. Therefore unless the error distribution is given, it's usually not easy to obtain the error spectrum. To alleviate this difficulty, two approximation algorithms are proposed. One is the Gaussian mixture method, which calculated the error spectrum by capturing the probability density function. The other using the sample is the power means error method. Furthermore, how the Gaussian mixture method and power means error method can be used in estimation performance evaluation are analyzed not only in the large sample case but also in the small sample case. Numerical examples are provided to illustrate the effectiveness of the above two algorithms. It is shown that the two proposed algorithms can be applied easily to calculate the error spectrum in estimator performance evaluation.

1. Introduction

In recent years, estimation performance evaluation (EPE) has received a great amount of attention due to their increasing use in estimation/filtering (see e.g., [1–5]), fusion and target tracking (see e.g., [6–8]). As pointed out in [9], a key aspect in EPE is the selection and proper interpretation of the metrics used for measuring the performance and determining the characteristics of the algorithms. So many new metrics have been provided in X.R. Li's work (see e.g., [1,9–12]), such as root mean square error (RMSE), average Euclidean error (AEE), harmonic average error (HAE), Geometric average error (GAE), iterative mid-range error (IMRE), median error and error mode.

Unfortunately, all of the above metrics can only reflect one aspect of the estimation performance. Thus, three comprehensive performance measures, i.e., error spectrum (ES), desirability level, and relative concentration and deviation measures were proposed in Refs. [13–15]. Among these metrics, ES can reveal more information about the estimation because it is an aggregation of many incomprehensive metrics. In addition, ES has a large array of important properties for performance evaluation, which can be found in Refs. [14,17].

However, ES has some limitations and drawbacks. On one hand, for dynamic systems, it is not easy to tell which is better since ES is a three-dimension plot for the whole time horizon. For this reason, dynamic error spectrum (DES) was proposed to solve this problem in Refs. [18,19]. However, DES is in essence the average height of error spectrum, which combines the error spectrum at a time instant into a single point. Thus this many to one mapping suffers from information loss. To overcome this, a new metric called enhanced error spectrum is proposed [20]. On the other hand, ES is difficult to calculate since it is defined by the error distribution which is unfortunately almost never available. To alleviate this difficulty, the Mellin transform provided a means to compute ES analytically [21]. Although ES is easy to obtain by using the Mellin transform, it is still required to know the error distribution. Nevertheless, in practical applications, it is actually hard to represent the error distribution but to obtain the sampling of the error. Thus, how to calculate ES using the sample is a worthwhile problem.

In this paper, the Gaussian mixture method and power means error method are proposed to calculate error spectrum approximately in EPE. Then how the Gaussian mixture method and power means error method can be used to evaluate an estimator's performance are discussed. As shown in Fig. 1, in large sample case, the Gaussian mixture method is in fact approximate to the ground error distribution by using the samples. While the power means

* Corresponding author. Tel.: +86 029-8478-7589.
E-mail addresses: peng_weishi@163.com (W.-S. Peng), ywfang2008@sohu.com (Y.-W. Fang), zhanrenjun@aliyun.com (R.-J. Zhan), wu_youli@126.com (Y.-L. Wu).

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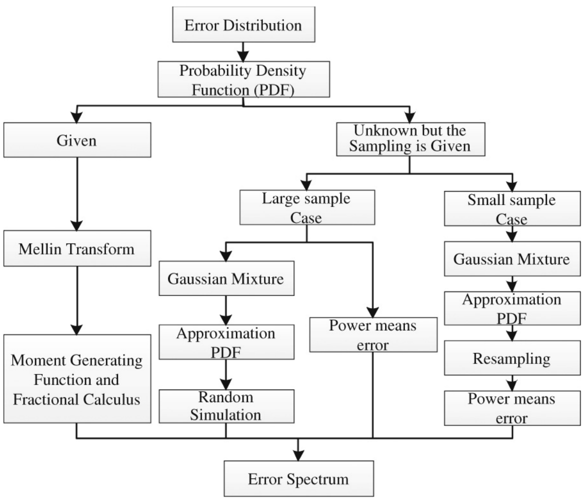

Fig. 1. Computation methods of error spectrum.

error method is straightforward to calculate ES based on the sampling. Whereas, in small sample case, the ground error distribution is approximately replace by the Gaussian mixture, which is used to resampling. And then the power means error method is applied to calculate the error spectrum based on the resampling samples.

This paper is organized as follows. ES is summarized in Section 2. Then the Gaussian mixture method and power means error method are proposed in Section 3. In Section 4, numerical examples are provided to illustrate the superiority of the above two approximation algorithms. Section 5 concludes this paper.

2. Summary of error spectrum

According to [13,14], given (vector-valued) estimator error $\tilde{\theta}$ of a (point) estimator $\hat{\theta}$, i.e., $\tilde{\theta} = \theta - \hat{\theta}$ where θ is the estimand (quantity to be estimated). Denote $e = ||\tilde{\theta}||$ or $e = ||\tilde{\theta}||/||\theta||$ as the absolute or relative estimation error norm, where $||\cdot||$ can be 1-norm or 2-norm. Then, for $r \in [-\infty, +\infty]$, ES is defined as

$$S(r) = [E(e^r)]^{1/r} = \left[\int e^r \mathrm{d}F(e)\right]^{1/r} = \begin{cases} \left[\int e^r f(e)\mathrm{d}e\right]^{1/r} & \text{if } e \text{ is continuous} \\ \left[\sum p_i e_i{}^r\right]^{1/r} & \text{if } e \text{ is discrete} \end{cases} \tag{1}$$

where $F(e)$, $f(e)$ and p_i are the cumulative distribution function (CDF), probability density function (PDF) and probability mass function (PMF), respectively.

From Eq. (1), it is clear that ES include many incomprehensive metrics as special cases by setting r to some specific values:

(a) $S(2) = \left(E\left[e^2\right]\right)^{1/2}$. Thus for a discrete e_i, $S(2)$ = RMSE.
(b) $S(1) = E[e]$. Thus for a discrete e_i, $S(1)$ = AEE.
(c) $S(0) \triangleq \lim_{r\to 0} S(r) = \exp(E[\ln e])$. Thus for a discrete e_i, $S(0) \triangleq$ GAE.
(d) $S(-1) = \left(1/E\left[1/e\right]\right)$. Thus for a discrete e_i, $S(-1)$ = HAE.

In view of this, the notation r used in this paper is the real number satisfying $r \in [-1, 2]$.

In application, $f(e)$ may not be available, that is, the error distribution is hard to represent, which lead to the difficulty computation of the error spectrum. Certainly, if the PDF of e is given, a nice method called Mellin transform have been proposed to calculate ES [21]:

$$M[f(e); r] = \int_0^\infty e^{r-1} f(e)\mathrm{d}e \tag{2}$$

By Eqs. (1) and (2), we have

$$S(r) = \left\{M[f(e); r+1]\right\}^{1/r} \tag{3}$$

Obviously, ES of many popular distributions can be obtained based on the transform pairs and properties of the Mellin transform [21]. Here is an illustrative example. Assuming that $e = \{e_i\}_{i=1}^n$ are follows the Rayleigh distribution, that is

$$f(e) = \frac{e}{k^2}\exp\left(\frac{-e^2}{2k^2}\right) \tag{4}$$

where $k > 0$ is the degree of freedom and $\exp(\cdot)$ is the Exponential function.

Substituting Eq. (4) into Eq. (3) yields

$$S(r) = \left\{M[f(e); r+1]\right\}^{1/r} = \left\{\int_0^\infty e^r \frac{e}{k^2}\exp\left(\frac{-e^2}{2k^2}\right)\mathrm{d}e\right\}^{1/r} \tag{5}$$

Let $u = e^2/2k^2$, then $e = k\sqrt{2u}$, thus Eq. (5) can be rewritten as

$$\begin{aligned} S(r) &= \left\{\int_0^\infty \frac{e^{r+1}}{k^2}\exp\left(\frac{-e^2}{2k^2}\right)\mathrm{d}e\right\}^{1/r} \\ &\overset{u=e^2/2k^2}{=} \left\{\int_0^\infty \frac{(k\sqrt{2u})^{r+1}}{k^2}\frac{k}{\sqrt{2u}}\exp(-u)\mathrm{d}u\right\}^{1/r} \\ &= \left\{(k\sqrt{2})^r\int_0^\infty u^{((r/2)+1)-1}\exp(-u)\mathrm{d}u\right\}^{1/r} \\ &= \sqrt{2}k\left\{\Gamma\left(\frac{r}{2}+1\right)\right\}^{1/r} \end{aligned} \tag{6}$$

where

$$\Gamma\left(\frac{r}{2}+1\right) = \int_0^\infty u^{((r/2)+1)-1}\exp(-u)\mathrm{d}u, \quad \frac{r}{2}+1 > 0$$

which is the Gamma function.

To supplement, if the PDF of e is given whereas the parameters are unknown, $f(e)$ can be estimated according to a sample of e. In fact, the error distribution is usually difficult to represent but to obtain the sampling. Thus, two new approximation algorithms are proposed next to calculate ES based on the samples.

3. Two approximation algorithms of error spectrum

Now, the Gaussian mixture method and power means error method are proposed to compute the error spectrum. The Gaussian mixture method is in fact approximate to the PDF of e by using the samples $\{e_i\}_{i=1}^n$, which can effectively capture any PDF as closely as desired. In this way, in Eq. (1), the PDF of e is replace by the Gaussian mixture PDF. Certainly this approximation leads to the difficulty computation for ES. Thus, the random simulation is applied to calculate the integral of ES. Another approximation algorithm is the power means error method, which is straightforward to calculate ES based on the sampling.