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Elegant deduction of three types of recursive convolution (RC) technique in simulation of dispersive material

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ABSTRACT

Recursive convolution (RC) method is widely used in simulation of dispersive material in finite-difference time-domain (FDTD). The method uses inverse Fourier transform to change the constitutive relation between polarization and electric field in frequency-domain to convolution integral form in time domain. The convolution is updated by the recursive method. This paper presents the elegant deduction of the RC method and shows that constant RC (CRC), piecewise linear RC (PLRC) and trapezoidal RC (TRC) are all special cases of a general form. The difference is due to the approximation of electric field in convolution integral. The numerical examinations validate the accuracy and efficiency of the algorithms.

1. Introduction

Finite-difference time-domain (FDTD) has been a very important method in computing electromagnetic problems including scattering, radiation and propagation since Yee proposed staggered grid and frog-step algorithm in 1966 [1]. Many techniques to extend the FDTD's capability have developed such as absorbing boundary condition (ABC) [2], total field/scattering field (TF/SF) division, far field exterior [3], FDTD's grid in general nonorthogonal coordinates [4] and perfectly matched layer (PML) [5,6]. Stability analysis was investigated in [7]. The FDTD is popular because it is very convenient for modeling of spatial distribution of materials and any shape of targets. It is a time domain method which can obtain spectral results by a single run.

The classical FDTD can only solve the nondispersive material, that is, the permeability, permittivity and conductivity must be constant. But, in real world, many materials have obvious dispersive property, such as sea water, muscle, organs and so on. The FDTD is very powerful in handling these kinds of dispersive materials. Generally, there are three methodologies in dealing with dispersive materials, that is, recursive convolution (RC) [8,9], auxiliary differential equation (ADE) [10] and Z-transform (ZT) [11]. The RC method expresses the constitutive relation between polarization and electric field in a convolution integral form which can be updated by recursion. The ADE method introduces auxiliary differential equation to update electric field. The ZT method transforms the constitutive relations to Z-domain and uses the time shifter to transform the Z-domain to discrete time domain.

The RC method is first proposed by Luebbers in 1990 in studying the characteristics of Debye media [12]. The classical RC assumed the electric field in the convolution integral is constant, so this RC method is also called constant RC (CRC). In 1991, Luebbers used the CRC method to study the reflection and transmission coefficients of plasma slab and obtained correct results [9]. Piecewise linear RC (PLRC) makes linear approximation of the electric field in the convolution integral yielding more accurate results [13]. Trapezoidal RC (TRC) technique provides the accuracy comparable to the PLRC, while requiring only one convolution integral as in the RC technique [14].

This paper represents a general RC formulation in the convolution integral. By introducing different assumptions of electric field in the interval, we obtain the CRC, the PLRC and the TRC of the general RC method. Numerical studies of the plasma slab's reflection and transmission coefficients using these RC methods show that the PLRC and the TRC have more accurate results than the CRC.

The time dependence e^{jwt} is assumed and suppressed in this paper.

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2. Formulation of RC method

2.1. General RC method

Dispersive materials are often modeled as

$$\varepsilon_r(\omega)=\varepsilon_\infty+\chi(\omega) \tag{1}$$

where, ω is angular frequency, ε_r is relative permittivity, ε_∞ is relative permittivity of infinite frequency and χ is electric susceptibility, which tends to zero when frequency gets infinite. To emphasize the frequency dependence of ε_r and χ, we keep argument ω in Eq. (1). Electric displacement is expressed as

$$\mathbf{D}(\omega)=\varepsilon_0(\varepsilon_\infty+\chi(\omega))\mathbf{E}(\omega) \tag{2}$$

where, ε_0 is permittivity in free space and $\mathbf{E}$ is electric field. In the RC method, the differential of $\mathbf{D}$ is discretized in central time difference form, that is

$$\frac{\partial \mathbf{D}}{\partial t}\approx\frac{\mathbf{D}^{n+1}-\mathbf{D}^n}{\Delta t} \tag{3}$$

where, Δt is time increment. The time domain constitutive relation after inverse Fourier's transform can be expressed as convolution form, that is

$$\mathbf{D}(t)=\varepsilon_0\left(\varepsilon_\infty\mathbf{E}(t)+\int_0^t\chi(t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\right) \tag{4}$$

The discretization of $\mathbf{D}$ at integer time step is

$$\mathbf{D}^n=\varepsilon_0\left(\varepsilon_\infty\mathbf{E}^n+\int_0^{n\Delta t}\chi(n\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\right) \tag{5}$$

The difference of $\mathbf{D}^n$ can be expressed as

$$\begin{aligned}\mathbf{D}^{n+1}-\mathbf{D}^n &= \varepsilon_0\varepsilon_\infty\left(\mathbf{E}^{n+1}-\mathbf{E}^n\right)\\ &\quad+\varepsilon_0\int_0^{(n+1)\Delta t}\chi((n+1)\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\\ &\quad-\varepsilon_0\int_0^{n\Delta t}\chi(n\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\\ &=\varepsilon_0\varepsilon_\infty\left(\mathbf{E}^{n+1}-\mathbf{E}^n\right)\\ &\quad-\varepsilon_0\int_0^{n\Delta t}(\chi(n\Delta t-\tau)-\chi((n+1)\Delta t-\tau))\mathbf{E}(\tau)\mathrm{d}\tau\\ &\quad+\varepsilon_0\int_{n\Delta t}^{(n+1)\Delta t}\chi((n+1)\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\left(\mathbf{E}^{n+1}-\mathbf{E}^n\right)\\ &\quad-\varepsilon_0\boldsymbol{\psi}^n+\varepsilon_0\mathbf{I}_\chi^n\end{aligned} \tag{6}$$

where,

$$\boldsymbol{\psi}^n=\int_0^{n\Delta t}\Delta\chi(n\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau \tag{7}$$

$$\mathbf{I}_\chi^n=\int_0^{\Delta t}\chi(\tau)\mathbf{E}((n+1)\Delta t-\tau)\mathrm{d}\tau \tag{8}$$

$$\Delta\chi(t)=\chi(t)-\chi(t+\Delta t) \tag{9}$$

Table 1
Coefficients of different RC methods.

	CRC	PLRC	TRC
A_x	χ_0	$\chi_0-\xi_0$	$\chi_0/2$
B_x	0	$-\xi_0$	$-\chi_0/2$
$A_{\Delta x}$	$\Delta\chi_0$	$\Delta\chi_0-\Delta\xi_0$	$\Delta\chi_0/2$
$B_{\Delta x}$	0	$-\Delta\xi_0$	$-\Delta\chi_0/2$

$\boldsymbol{\psi}^n$ can be calculated by recursive convolution which is expressed as

$$\begin{aligned}\boldsymbol{\psi}^{n+1}&=\int_0^{(n+1)\Delta t}\Delta\chi((n+1)\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\\ &=\int_0^{n\Delta t}\Delta\chi((n+1)\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\\ &\quad+\int_{n\Delta t}^{(n+1)\Delta t}\Delta\chi((n+1)\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau\\ &=\int_0^{n\Delta t}C_{\Delta\chi}(n\Delta t-\tau)\Delta\chi(n\Delta t-\tau)\mathbf{E}(\tau)\mathrm{d}\tau+\mathbf{I}_{\Delta\chi}^n\end{aligned} \tag{10}$$

where,

$$\mathbf{I}_{\Delta\chi}^n=\int_0^{\Delta t}\Delta\chi(\tau)\mathbf{E}((n+1)\Delta t-\tau)\mathrm{d}\tau \tag{11}$$

$$C_{\Delta\chi}(t)=\Delta\chi(t+\Delta t)/\Delta\chi(t) \tag{12}$$

For Debye media, $\chi(\omega)=(\varepsilon_s-\varepsilon_\infty)/(1+j\omega\tau_0)$ and the time domain form is $\chi(t)=(\varepsilon_s-\varepsilon_\infty)/\tau_0e^{-t/\tau_0}$. We get $\Delta\chi(t)=(\varepsilon_s-\varepsilon_\infty)/\tau_0e^{-t/\tau_0}(1-e^{-\Delta t/\tau_0})$ and see that $C_{\Delta\chi}(t)=e^{-\Delta t/\tau_0}$ is a constant.

For Drude media, $\chi(\omega)=\omega_p^2/j\omega(\nu+j\omega)$ and the time domain form is $\chi(t)=\frac{\omega_p^2}{\nu}\left(1-e^{-\nu t}\right)$. We get $\Delta\chi(t)=-\frac{\omega_p^2}{\nu}e^{-\nu t}\left(1-e^{-\nu\Delta t}\right)$ and see that $C_{\Delta\chi}(t)=e^{-\nu\Delta t}$ is a constant.

We can see that for Debye and Drude media, $\Delta\chi$ has exponential form and $C_{\Delta\chi}(t)$ is a constant, which still denoted as $C_{\Delta\chi}$ for less variables. For these two media, $\boldsymbol{\psi}^{n+1}$ can be expressed in a recursive form of

$$\boldsymbol{\psi}^{n+1}=C_{\Delta\chi}\boldsymbol{\psi}^n+\mathbf{I}_{\Delta\chi}^n \tag{13}$$

Up till now, we have not made any assumptions in convolution integral, so Eqs. (6) and (13) are suitable for any media that $\Delta\chi$ is exponentially time variant. We can see that the key role in the recursion is $\mathbf{I}_\chi^n$ and $\mathbf{I}_{\Delta\chi}^n$. The integral interval is [0, Δt], the length of which is one time increment. The integrand is multiplication of $\chi(\tau)$ or $\Delta\chi(\tau)$ and $\boldsymbol{E}((n+1)\Delta t-\tau)$ whose values in the terminal of integral interval is $\mathbf{E}^{n+1}$ and $\mathbf{E}^n$. In the FDTD calculations, we did not know the exact value of the electric field over time interval $[n\Delta t,(n+1)\Delta t]$ because the fields are only sampled at integer time step. The integral $\mathbf{I}_\chi^n$ and $\mathbf{I}_{\Delta\chi}^n$ can be approximated by $\mathbf{E}^{n+1}$ and $\mathbf{E}^n$. We assume that $\mathbf{I}_\chi^n$ and $\mathbf{I}_{\Delta\chi}^n$ are linear combination of $\mathbf{E}^{n+1}$ and $\mathbf{E}^n$, that is

$$\begin{aligned}\mathbf{I}_\chi^n&=A_\chi\mathbf{E}^{n+1}-B_\chi\mathbf{E}^n\\ \mathbf{I}_{\Delta\chi}^n&=A_{\Delta\chi}\mathbf{E}^{n+1}-B_{\Delta\chi}\mathbf{E}^n\end{aligned} \tag{14}$$

where, $A_x, B_x, A_{\Delta x}, B_{\Delta x}$ are constants. Considering Ampere's law in source free-space $\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}$ and Eqs. (3) and (6), we obtain the general form of the RC method as follows:

$$\mathbf{E}^{n+1}=\frac{\varepsilon_\infty+B_\chi}{\varepsilon_\infty+A_\chi}\mathbf{E}^n+\frac{\boldsymbol{\psi}^n}{\varepsilon_\infty+A_\chi}+\frac{1}{\varepsilon_\infty+A_\chi}\frac{\Delta t}{\varepsilon_0}\nabla\times\mathbf{H}^{n+1/2} \tag{15}$$

$$\psi^{n+1}=C_{\Delta\chi}\psi^n+A_{\Delta\chi}\mathbf{E}^{n+1}-B_{\Delta\chi}\mathbf{E}^n \tag{16}$$