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New convolution and product theorem for the linear canonical transform and its applications

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ABSTRACT

The convolution theorem for the linear canonical transform (LCT) is of importance in signal processing theory and application. Recently, some attempts at extending the convolution theorem in the Fourier transform (FT) domain to the LCT domain have derived many important results, which are very useful and effective in filter design and signal reconstruction, but none of them generalize very nicely and simply the classical result for the FT. In this paper, we formulate a new kind of convolution structure for the LCT, which has the elegance and simplicity in both time and LCT domains comparable to that of the FT and preserves the commutative and associative properties. Then with the new convolution theorem, it is easy to implement in the designing of multiplicative filters through both the new convolution in the time domain and the product in the LCT domain, and it is convenient to deduce the Shannon-type reconstruction formula for bandlimited signals in the LCT domain. Theoretical analyses and numerical simulations are also presented to show the correctness and effectiveness of the proposed techniques.

1. Introduction

The convolution and product theorems play a significant role in many fields of signal processing [1]. It is clear that the classical convolution theorem for the Fourier transform (FT) is very convenient and powerful for the designing of multiplicative filters in the FT domain. It is given by [1]

$$f(t)*g(t)=(f*g)(t)=\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\mathrm{d}\tau, \tag{1}$$

$$(f*g)(t)\stackrel{FT}{\longleftarrow}F(u)G(u), \tag{2}$$

where * stands for the conventional convolution operation in the time domain, and $F(u)$ and $G(u)$ represent the FTs of the signals $f(t)$ and $g(t)$, respectively. As shown in (1) and (2), this theorem exhibits the following characteristics and applications:

(i) In the time domain, the convolution of two signals takes a single integral expression, and then the designed filters can be achieved easily through the convolution in the time domain.
(ii) In the transformation (FT) domain, the expression takes a simple multiplication of the signals' transformations (FTs), and then the multiplicative filters can be achieved easily through the product in the transformation (FT) domain.
(iii) The convolution exhibits the commutative property, which sees great use in digital signal processing for manipulating equations [2].
(iv) The convolution exhibits the associative property, which is used in system theory to describe how cascaded systems behave, that is, any number of cascaded systems can be replaced with a single system [2].

Moreover, based on this theorem, the classical Shannon sampling theorem can be deduced expediently [3].

The linear canonical transform (LCT) is a class of linear integral transform with three free parameters [4–11]. It can be considered as a quadratic phase system (QPS), which is one of the most important optical systems and is implemented with an arbitrary number of thin lenses and propagation through free space in the Fresnel approximation or through sections of graded-index media [12]. Then it can be defined as the output light field of the QPS [13].

$$F_A(u)=L_A[f(t)](u)=\begin{cases}\displaystyle\int_{-\infty}^{+\infty} f(t)K_A(u,t)\mathrm{d}t, & b\neq 0\\ \sqrt{d}e^{j\frac{cd}{2}u^2}f(du), & b=0\end{cases}, \tag{3}$$

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where

$$K_A(u,t)=\frac{1}{\sqrt{j2\pi b}}e^{j\left(\frac{d}{2b}u^2-\frac{1}{b}ut+\frac{a}{2b}t^2\right)} \tag{4}$$

is the LCT kernel with parameter matrix $A=(a,\ b;c,\ d)$ and the parameters a, b, c, d are real numbers satisfying $ad-bc=1$. Some well-known signal analysis tools, for instance, the FT, the fractional Fourier transform (FRFT), the Fresnel transform (FST), and the scaling operations are all special cases of the LCT [14]. It was applied to solve the differential equations and analyze the optical systems and has recently attracted much attention due to its preponderance for processing non-stationary signals. The LCT also has found many applications in optics, pattern recognition, filter design, and radar and sonar systems analysis [15–18]. With in-depth research on it, many important theories and concepts in the FT and FRFT domains have been extended into the LCT domain, including the convolution and product theorems [19–27].

Deng et al. firstly discussed the convolution and product theorems for the LCT and proposed a convolution structure on the basis of the conventional convolution directly [19]. It takes the form

$$(f*g)(t)\overset{LCT}{\longleftrightarrow}\frac{1}{|a|}e^{j\frac{c}{2a}u^2}\int_{-\infty}^{+\infty}F_A(v)e^{-j\frac{c}{2a}v^2}g\left(\frac{u-v}{a}\right)\mathrm{d}v. \tag{5}$$

Compared with the classical result in the FT domain, this definition of convolution theorem satisfies (i), (iii) and (iv), but (ii) does not hold since it is complicated to reduce the right-hand-side of (5) to a simple multiplication of the signals' LCTs. Therefore, it is inconvenient for the discussion of multiplicative filters in the LCT domain by use of (5). In [19], Deng et al. formulated another convolution theorem for the LCT to address the above constraint. Here, we show it below:

$$(f\overset{A}{*}g)(t)=\frac{1}{\sqrt{j2\pi b}}e^{-j\frac{a}{2b}t^2}((f(t)e^{j\frac{a}{2b}t^2})*(g(t)e^{j\frac{a}{2b}t^2})), \tag{6}$$

$$L_A[(f\overset{A}{*}g)(t)](u)=F_A(u)G_A(u)e^{-j\frac{d}{2b}u^2}, \tag{7}$$

where $\overset{A}{*}$ represents the modified convolution operation for the LCT. Wei et al. also derived this theorem from the point of view of rewriting the modified convolution expression as a simple one-dimensional integral and defining a convolution operation associated with a weight function, respectively [20]. It can be seen from a single integral reproduced here [20,21],

$$(f\overset{A}{*}g)(t)=\frac{1}{\sqrt{j2\pi b}}\int_{-\infty}^{+\infty}f(\tau)g(t-\tau)e^{j\frac{a}{b}\tau(\tau-t)}\mathrm{d}\tau, \tag{8}$$

where $e^{j\frac{a}{b}\tau(\tau-t)}$ is a so-called weight function. From (7), although the multiplicative filters can be achieved theoretically through the product in the LCT domain, there exists an extra chirp multiplier which imposes difficulty in real applications since it is nearly impossible to generate a chirp signal accurately in practical engineering. For this, two kinds of convolution structures for the LCT were presented to eliminate the chirp multiplier in the right-hand-side of (7).

To be specific, according to (1) and (2) and the relationship between the LCT and FT, Wei et al. [22] introduced a new convolution structure which has the form [22–24]

$$\begin{aligned}(f\overset{A}{\tilde{*}}g)(t)&=e^{-j\frac{a}{2b}t^2}((f(t)e^{j\frac{a}{2b}t^2})*(g(t)))\\&=\int_{-\infty}^{+\infty}f(\tau)g(t-\tau)e^{j\frac{a}{2b}(\tau^2-t^2)}\mathrm{d}\tau,\end{aligned} \tag{9}$$

$$L_A[(f\overset{A}{\tilde{*}}g)(t)](u)=F_A(u)G\left(\frac{u}{b}\right), \tag{10}$$

where $\overset{A}{\tilde{*}}$ denotes the further-modified convolution operation. In reality, this structure of convolution was first introduced by Shi et al. in [24] in terms of the FRFT, a special case of the LCT. This further-modified convolution is equivalent to the definition given by (6)–(8) but contains lesser chirp multipliers, and hence it is easier to implement in filter design through both convolution in the time domain and product in the LCT domain. Meanwhile, the further-modified convolution exhibits the distributive property, which describes the operation of parallel systems with added outputs and allows this kind of systems to be replaced with a single system, playing a fundamental role in multichannel sampling and reconstruction. However, this convolution does not exhibit the commutative and associative properties, and therefore, the applications performed in (iii) and (iv) are subjected to certain restrictions. Moreover, from mathematics point of view this new definition does not parallel with the classical result for the FT because of non-symmetric multiplication expression in the LCT domain, as implied by the right-hand-side of (10). Another new convolution structure for the LCT obtained by Wei et al. preserves exact multiplication of the signals' LCTs in the LCT domain comparable to exact multiplication of the signals' FTs in the FT domain of the classical result, defining a generalized convolution operation $\overset{A}{\Theta}$ [25], i.e.,

$$\begin{aligned}&(f\overset{A}{\Theta}g)(t)\\&=\int_{-\infty}^{+\infty}f(\tau)g(t\theta\tau)\mathrm{d}\tau\\&=\frac{1}{2\pi|b|}\int_{-\infty}^{+\infty}f(\tau)e^{-j\frac{a}{2b}(t^2-\tau^2)}\left(\int_{-\infty}^{+\infty}G_A(u)e^{j\frac{1}{b}u(t-\tau)}\mathrm{d}u\right)\mathrm{d}\tau,\end{aligned} \tag{11}$$

$$(f\overset{A}{\Theta}g)(t)\overset{LCT}{\longleftrightarrow}F_A(u)G_A(u), \tag{12}$$

where $g(t\theta\tau)$ stands for the generalized translation for the LCT. As shown in (12), this generalized convolution theorem has the elegance and simplicity in the LCT domain comparable to the convolution theorem for the FT in the FT domain, and then it is very easy to implement in filter design through the product in the LCT domain. Meanwhile, the generalized convolution exhibits the commutative, associative and distributive properties, having a number of applications for manipulating equations, simplifying cascaded systems and analyzing multichannel systems. However, the generalized convolution expression is a triple integral form, and hence it is complicated to turn it into a single integral form, as implied by (11). Therefore, it causes heavy computational load to achieve filter design through the generalized convolution in the time domain.

In order to formulate a closed-form expression for LCT's convolution theorem, Shi et al. proposed a kind of unified convolution structure for the LCT by use of an unified canonical convolution operation Ξ_{A_1,A_2,A_3} [26], that is,

$$(f\Xi_{A_1,A_2,A_3}g)(t)=\int_{-\infty}^{+\infty}f(\tau)(\mathrm{T}_\tau^{A_1}g)(t)\phi_{a_1,a_2,a_3}(t,\tau)\mathrm{d}\tau, \tag{13}$$

$$L_{A_3}[(f\Xi_{A_1,A_2,A_3}g)(t)](u)=\epsilon_{d_1,d_2,d_3}(u)F_{A_1}\left(\frac{ub_1}{b_3}\right)G_{A_2}\left(\frac{ub_2}{b_3}\right). \tag{14}$$

Although this unified convolution theorem is a generalization of the classical convolution theorem for the FT and the modified and further-modified convolution theorems for the LCT, it lacks concinnity and simplicity and then has little advantage in applications. In reality, the authors merely considered the application of it in the