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Synchronization between different dimensional chaotic systems using two scaling matrices

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ABSTRACT

The epitome of this paper centers on chaos synchronization problem of different dimensional chaotic systems in different dimensions using two scaling matrices, the Lyapunov stability theory, and the stability theory of linear system. The controller is designed to assure that the synchronization of two different dimensional chaotic systems is achieved. Numerical examples and computer simulations are used to validate, numerically, the proposed synchronization schemes.

1. Introduction

Our natural world consists of physical systems that are undoubtedly nonlinear. It has been repeatedly demonstrated by scientists in the last recent decades that nonlinear systems, which models our real world, can display a variety of behaviors including chaos and hyperchaos. The most important characteristic of chaotic dynamics is its critical sensitivity to initial conditions, which is responsible for initially neighboring trajectories separating from each other exponentially in the course of time. This behavior made chaos undesirable and unwanted in many cases of research as it reduces their predictability over long time scales. But this special attribute may be a valuable advantage in certain areas of research. Chaotic dynamics has the ability to amplify small perturbations which improves their utility for reaching specific desired states with very high flexibility and low energy cost. In other words, we could try to control chaos for the benefit of our needs. Synchronization of different chaotic or hyperchaotic systems is one of the few main control methods popularly discussed recently. This is generally due to its prospective applications especially in chemical reactions, power converters, biological systems, information processing, secure communications, etc. [1]. The current problems of synchronization of chaos are very interesting, non-traditional, and indeed very challenging [2,3]. A wide variety of approaches have been proposed for chaos synchronization such as adaptive control [4–9], linear and nonlinear feedback control [10–16], active control [17–21] complete synchronization [22] and projective synchronization [24–28]. To the best of our knowledge, most of the existing papers discuss the synchronization between two chaotic or hyperchaotic with the same dimension. However, in many real physics systems, the synchronization is carried out through the oscillators with different dimensions, especially the systems in biological science and social science. For example, in the cardiorespiratory system, the synchronization between the heart and the lung has been found even though their models have different dimensions [23]. In this paper, we proposed a method to synchronize two chaotic systems in different dimensions even though they have different dimensions, the synchronization controller is designed based on Lyapunov stability theory and the stability theory of linear system. An analytic expression of the controller is shown. Finally, illustrative examples of chaotic and hyperchaotic systems are used to show the effectiveness of the proposed method.

2. Theory

Consider the drive system in the form of

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in R^n$ is the state vector of the drive system (1), $f: R^n \to R^n$ defines a vector field in n-dimensional space. On the other hand, the response system is assumed by

$$\dot{y}(t) = g(y(t)) + U, \tag{2}$$

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where $y(t) \in R^m$ is the state vector of the response system (2), $g: R^m \to R^m$ defines a vector field in m-dimensional space, $U \in R^m$ is control input vector.

Definition 1. The drive system (1) and the response system (2) are said to be synchronized in dimension d, with respect to scaling matrices Θ and Φ, if there exists a controller $U \in R^m$ and given matrices $\Theta = (\Theta)_{d\times m}$ and $\Phi = (\Phi)_{d\times n}$ such that the synchronization error $e(t) = \Theta y(t) - \Phi x(t)$, satisfies that $\lim_{t\to\infty} \|e(t)\| = 0$.

2.1. Synchronization between 3D drive and 4D response chaotic systems in 3D

In order to observe the synchronization behavior between between 3*D* drive and 4*D* response chaotic system in 3*D*, the drive and the response systems are defined below respectively,

$$\dot{x}(t) = Ax(t) + f(x(t)), \tag{3}$$

where $x(t) \in R^3$ is state vector, $A \in R^{3\times3}$, $f: R^3 \to R^3$ are the linear part and the nonlinear part of system (3), respectively. Eq. (3) is considered as a drive system. By introducing an additive control $U = (u_1, u_2, u_3, u_4)^T \in R^4$, then the controlled response system is given by

$$\dot{y}(t) = g(y(t)) + U, \tag{4}$$

where $y(t) \in R^4$ is state vector, $g: R^4 \to R^4$. The error system between the drive system (3) and the response system (4), can be derived as

$$\dot{e}(t) = (A - L_1)e(t) + R + \Theta U, \tag{5}$$

where $\Theta = (\Theta_{ij}) \in R^{3\times4}$, $\Phi = (\Phi_{ij}) \in R^{3\times3}$ are scaling matrices and

$$R = (L_1 - A)\Theta y(t) - ((L_1 - A)\Phi + \Phi A)x(t) + \Theta g(y(t)) - \Phi f(x(t)), \tag{6}$$

and $L_1 \in R^{3\times3}$ is an unknown control matrix to be determined. Assume that $U = (u_1, u_2, u_3, 0)$. Then, the error system (5), can be written as

$$\dot{e}(t) = (A - L_1)e(t) + R + \hat{\Theta}\hat{U}, \tag{7}$$

where

$$\hat{\Theta} = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{21} & \Theta_{22} & \Theta_{21} \\ \Theta_{31} & \Theta_{32} & \Theta_{33} \end{pmatrix}, \tag{8}$$

and $\hat{U} = (u_1, u_2, u_3)^T$, is the new control law. To achieve synchronization between systems (3) and (4), the controller $\hat{U}$ is chosen as

$$\hat{U} = -\hat{\Theta}^{-1}R, \tag{9}$$

where $\hat{\Theta}^{-1}$ is the inverse of $\hat{\Theta}$. By substituting Eq. (9) in Eq. (5), the error system can be described as

$$\dot{e}(t) = (A - L_1)e(t). \tag{10}$$

Theorem 1. *If there exists a positive definite matrix P, such that*

$$(A - L_1)^T + (A - L_1) = -P. \tag{11}$$

Then, the drive system (3) *and the response system* (4) *are globally synchronized, with respect to scaling matrices* Θ *and* Φ*, under the controller* (9).

Proof. Construct the candidate Lyapunov function in the form

$$V(e(t)) = e^T(t)e(t), \tag{12}$$

then the time derivative of V along the solution of error dynamical system equation (10) gives that

$$\begin{aligned}\dot{V}(e(t)) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\ &= \dot{e}^T(t)(A - L_1)^T e(t) + e^T(t)(A - L_1)e(t) \\ &= e^T(t)[(A - L_1)^T + (A - L_1)]e(t) = -e^T(t)Pe(t) < 0.\end{aligned}$$

It is clear that V is positive definite and $\dot{V}$ is negative definite in the neighborhood of the zero solution for system (10). Therefore, the systems (3) and (4) are globally synchronized asymptotically, i.e $\lim_{t\to\infty} \|e(t)\| = 0$. This completes the proof. □

2.2. Synchronization between 3D drive and 4D response chaotic systems in 4D

In order to observe the synchronization behavior between between 3*D* drive and 4*D* response chaotic system in 4*D*, the drive and the response systems are defined below, respectively,

$$\dot{x}(t) = f(x(t)), \tag{13}$$

where $x(t) \in R^3$ is state vector, $f: R^3 \to R^3$. Eq. (13) is considered as a drive system. By introducing an additive control $U = (u_1, u_2, u_3, u_4)^T \in R^4$, then the controlled response system is given by

$$\dot{y}(t) = By(t) + g(y(t)) + U, \tag{14}$$

where B is an 4×4 constant matrix, $g: R^4 \to R^4$ is a nonlinear function and $U \in R^4$ is a controller. The error system between the drive system (13) and the response system (14), can be derived as

$$\dot{e}(t) = (B - L_2)e(t) + R + \Theta U, \tag{15}$$

where $\Theta = (\Theta_{ij}) \in R^{4\times4}$, $\Phi = (\Phi_{ij}) \in R^{4\times3}$ are scaling matrices, and

$$R = ((B - L_2)\Theta + \Theta B)y(t) - (B - L_2)\Phi x(t) + \Theta g(y(t)) - \Phi f(x(t)), \tag{16}$$

where $L_2 \in R^{4\times4}$ is an unknown control matrix to be determined. To achieve synchronization between systems (13) and (14), we choose the controller U as

$$U = -\Theta^{-1}R, \tag{17}$$

where Θ^{-1} is the inverse matrix of Θ.

Theorem 2. *If* L_2 *is chosen such that all eigenvalues of* $B - L_2$ *are strictly negative, then the drive system* (13) *and the response system* (14) *are globally synchronized with respect to* Θ *and* Φ*, under the control law* (17).

Proof. By substituting Eq. (17) in Eq. (15), the error system can be written as

$$\dot{e}(t) = (B - L_2)e(t). \tag{18}$$

According to the stability criterion of linear system, if all eigenvalues of $B - L_2$ are strictly negative, it is immediate that all solution of error system (18) go to zero as $t \to \infty$. Therefore, systems (13) and (14) are globally synchronized. This completes the proof. □

3. Applications

In this section, we give two examples to show the effectiveness of our proposed synchronization schemes. We choose the Rössler