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Multiple attractors and robust synchronization of a chaotic system with no equilibrium

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ABSTRACT

This paper introduces a three-dimensional system with no equilibrium point in which the Shilnikov method is not applicable to demonstrate the chaos. The remarkable particularity of the system is that it can generate multiple attractors with different system parameters and initial values. To further understand the complex dynamics, some basic properties of the system are studied theoretically and numerically. Simultaneously, by considering the sensibility to system parameter and initial value, a robust synchronization scheme of this chaotic system is proposed. Sufficient conditions to guarantee synchronization are given in the sense of H_∞ stability theory, numerical simulations are performed to further verify the effectiveness.

1. Introduction

Since the famous Lorenz system was found in 1963 [1], numerous chaotic systems have been reported [2–9]. For three-dimensional autonomous hyperbolic systems, a generally recognized method proving the existence of chaos is Shilnikov criterion, which brings on four classes of chaos: homoclinic orbit chaos, heteroclinic orbit chaos, hybrid type with both homoclinic and heteroclinic orbits, and chaos without homoclinic orbit and heteroclinic orbit. Recall that a homoclinic orbit is a trajectory that is doubly asymptotic to an equilibrium point, or closed orbit asymptotic to itself. A heteroclinic orbit is a trajectory that connects an equilibrium point to another, or connects a closed orbit to another closed orbit, respectively.

Recently, several researchers direct toward finding and analyzing peculiar chaotic flows with stable equilibrium or a line equilibrium, or in which there exists no equilibrium at all [10–18]. For such systems, there is little knowledge about the dynamical properties, and the Shilnikov criteria cannot be used to verify the chaos because of the lack of homoclinic or heteroclinic orbit [10]. This further reveals that the analytic criterion about which the system has at least one unstable equilibrium point for generating chaos is certainly not necessary.

In this paper, we introduce a three-dimensional autonomous system with one quadratic cross-product term, one square term, and one constant term. Since there is no equilibrium point and thus there is no homoclinic or heteroclinic orbit in this dynamical system, the Shilnikov method is not applicable to verify the existence of chaos. The remarkable feature of the system is that it can generate multiple-shape attractors with different system parameters and different initial values, which greatly enhanced the potential application of this system in secure communications since the complexity of the dynamics can be flexibly chosen or well increased. To further understand the complex dynamics, some basic properties of the system are studied theoretically and numerically. Simultaneously, by considering the sensibility to system parameter and initial value, a H_∞ control scheme is proposed to realize complete synchronization of this chaotic system, the control scheme is simple with a single input and robust against the uncertainty/disturbance of system parameter and initial value. Numerical simulations illustrate that the proposed control law is effective.

2. The proposed chaotic system

2.1. System description

In the search for chaotic flows with no equilibrium, the following unusual system was obtained:

$$\begin{cases} \dot{x}_1 = x_2 - a \\ \dot{x}_2 = -x_1 + x_2 x_3 \\ \dot{x}_3 = x_2 - {x_2}^2 \end{cases} \tag{1}$$

This is a one-parameter family of chaotic flows in the sense that a sequence of multiple attractors can be continuously observed as the real parameter a varies gradually, and the space of coefficients was

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Table 1
Dynamical properties of the system for some typical values of parameter a.

Case	Parameter a	Les	D_{KY}	Dynamical property
(a)	0.39	(0.043159, 0, −0.047389)	2.9107	Chaos
(b)	0.46	(0.025271, 0, −0.029027)	2.8706	Chaos
(c)	0.48	(0.016063, 0, −0.01683)	2.9544	Chaos
(d)	0.7	(0.0037816, 0, −0.0041671)	2.9075	Chaos

searched for the "elegant" values by which every other coefficients are set to ±1.

It is easy to know the invariance of the system under coordinate transformation $(x_1, x_2, x_3, t) \to (-x_1, x_2, -x_3, -t)$, which results in the occurrence of two types of solutions [16]. The first type has a 180° rotational symmetry about the x_2-axis and is time-reversal invariant. Thus it exhibits conservative behavior. The second type has one attractor in forward time and another attractor in reversed time, which are symmetric mutually under the rotation of 180° about the x_2-axis. The reversed time attractor is a repellor in forward time. Since it cannot be both an attractor and a repellor for the same points in state space, any symmetric solutions necessarily conserve state space volume on average. Thus, it brings on the emergence of both behaviors depending on the initial conditions.

2.2. Multiple attractors

One of distinct feature of the system is that it can generate strange attractors with different shape over a wide range of parameter. As the illustration, some typical values of parameter a with the same initial value (0.01, 0.03, −0.05) that lead to different system portraits are summarized in Table 1. All these cases are dissipative with $LE_1 + LE_2 + LE_3 < 0$ and the attractors projected onto the x_1x_3-plane are shown in Fig. 1.

Besides, from the information in Table 1, the Kaplan–Yorke dimension (D_{KY}) of the proposed system is closer to 3 and is much greater than most of the existed chaotic systems, specially compared with the chaotic flows with no equilibria introduced by [15] in which the Kaplan–Yorke dimension is only slightly greater than 2.

The third striking feature for the proposed system is that, with the variety of initial value $x_2(0)$, it is found to give rise to the coexistence of periodic, ribbon and strange attractors. We summarize below as an example with $a = 0.52$.

(1) The initial condition (0.01, 0.6, −0.05) gives a strange attractor with Lyapunov exponents (0.019456, 0, −0.021678) and a Kaplan–Yorke dimension of 2.8975, and the initial condition (0.01, −0.6, −0.05) gives a ribbon with Lyapunov exponents (0, 0, 0) and $D_{KY} = 2.0$. The phase portraits are demonstrated in Fig. 2(a) and (b), respectively.
(2) The initial condition (0.01, 1.3, −0.05) gives a strange attractor with Lyapunov exponents (0.01925, 0, −0.020134) and a Kaplan–Yorke dimension of 2.9561, and the initial condition (0.01, −1.3, −0.05) gives a periodic orbit with Lyapunov exponents (0, −0.036174, −0.7322) and $D_{KY} = 1.0$. The phase portraits are demonstrated in Fig. 3(a) and (b), respectively.
(3) The initial condition (0.01, 1.2, −0.05) gives a complex ribbon with LEs = (0, 0, 0) and $D_{KY} = 2.0$, and initial condition (0.01, −1.2, −0.05) gives a periodic orbit with LEs = (0, −0.036211, −0.73447) and $D_{KY} = 1.0$. The phase portraits are demonstrated in Fig. 4(a) and (b), respectively.

2.3. Analysis of equilibrium point

2.3.1. Case with a line equilibrium

Applying the equilibrium condition to system (1), it is found that there exists a line equilibrium point as $x_* = (0, 0, x_3)$ for $a = 0$, and the corresponding eigenvalues are $\lambda_1 = 0$, $\lambda_{2,3} = (x_3 \pm \sqrt{x_3{}^2 - 4})/2$. One holds the following statements.

(I) When $x_3 > 2$, we have $\lambda_{2,3} > 0$. So system (1) has a normally hyperbolic unstable node at equilibrium x_*.
(II) When $x_3 < -2$, we have $\lambda_{2,3} < 0$. So system (1) has a normally hyperbolic stable node at equilibrium x_*.
(III) When $-2 < x_3 < 0$, $\lambda_{2,3} = x_3 \pm i\sqrt{4 - x_3{}^2}$, $\mathrm{Re}(\lambda_{2,3}) < 0$. System (1) has a normally hyperbolic stable focus at equilibrium x_*.
(IV) When $0 < x_3 < 2$, $\lambda_{2,3} = x_3 \pm i\sqrt{4 - x_3{}^2}$, $\mathrm{Re}(\lambda_{2,3}) > 0$. System (1) has a normally hyperbolic unstable focus at equilibrium x_*.

If $a = 1$, it is found that there exists another line equilibrium point as $x_{**} = (x_3, 1, x_3)$ with eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = (x_3 \pm \sqrt{x_3{}^2 - 8})/2$. Similarly, the following statements hold.

(I) When $x_3 > 2\sqrt{2}$, we have $\lambda_{2,3} > 0$. System has a normally hyperbolic unstable node at equilibrium x_{**}.
(II) When $x_3 < -2\sqrt{2}$, we have $\lambda_{2,3} < 0$. System has a normally hyperbolic stable node at equilibrium x_{**}.
(III) When $-2\sqrt{2} < x_3 < 0$, $\lambda_{2,3} = x_3 \pm i\sqrt{8 - x_3^2}$, $\mathrm{Re}(\lambda_{2,3}) < 0$. System (1) has a normally hyperbolic stable focus at equilibrium x_{**}.
(IV) When $0 < x_3 < 2\sqrt{2}$, $\lambda_{2,3} = x_3 \pm i\sqrt{8 - x_3^2}$, $\mathrm{Re}(\lambda_{2,3}) > 0$. System (1) has a normally hyperbolic unstable focus at equilibrium x_{**}.

The system is the best-known example in which there exists a line equilibrium when $a = 0$ or $a = 1$, but it does not have a strange attractor since it is conservative.

2.3.2. Case with no equilibrium

If $a \neq 0$ and $a \neq 1$, it is found that there exists no equilibrium point in system (1) which does have strange attractors since the dissipative condition of $LE_1 + LE_2 + LE_3 \leq 0$ will cater. Even so, little knowledge about the dynamical properties is acquired. Recently, a new classification of self-excited attractor or hidden attractor for chaotic systems is introduced [18,19]. Self-excited attractor is associated with an unstable equilibrium that has lost its stability but that remains in its basin of attraction. Also a hidden attractor has a basin of attraction but it does not intersect with any small neighborhoods of equilibrium points. Obviously, chaotic systems with no equilibrium are part of examples of hidden attractors, which are rarely found but of theoretical and practical significance since they allow unexpected and potentially disastrous responses to any tiny perturbations.

3. Synchronization of the proposed chaotic system

As argued above, dynamical behavior of system (1) is of sensibility to system parameter and initial value. Actually, in practical synchronization process, disturbance of system parameter exists inevitably, and the initial values of synchronization systems cannot be predetermined, which will make the real synchronization problem much more complicated and even destroy the performance of synchronization. In this paper, a H_∞ control scheme is developed to realize complete synchronization of this chaotic system, the control scheme is simple with a single input yet robust against the uncertainties of system parameter and initial value.