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# Designing synchronization schemes for fractional-order chaotic system via a single state fractional-order controller

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ABSTRACT

In this paper the synchronization of fractional-order chaotic systems is studied and a new single state fractional-order chaotic controller for chaos synchronization is presented based on the Lyapunov stability theory. The proposed synchronized method can apply to an arbitrary three-dimensional fractional chaotic system whether the system is incommensurate or commensurate. This approach is universal, simple and theoretically rigorous. Numerical simulations of several fractional-order chaotic systems demonstrate the universality and the effectiveness of the proposed method.



## 1. Introduction

In recent years, fractional dynamic systems, which is the generalization of integer-order dynamic system, provide better mathematical models for some actual physical and engineering systems [1,2]. The fractional nonlinear dynamic systems have many dynamic behaviors which are similar to the integer-order systems, such as chaos, bifurcation, and attractor [3–6]. Control of the chaotic systems and chaotic phenomenon synchronization have potential applications in physics, biology, information, chemistry, and other fields [7–9]. But we find that chaos synchronization is very important but also very difficult. In this area, there are a few synchronization methods, such as PC control [10], the nonlinear state observer method [11], the sliding mode control [12] and adaptive-feedback control [13]. However, most of the above methods apply to certain fractional systems, and there are few general methods that can be applied to control arbitrary fractional chaotic systems.

On the other hand, the fractional-order control, which is a generalization of the traditional integer-order control, is becoming a matter of concern because of its flexibility and integrity [2,14–16]. The TID controller [14], the $PI^{\lambda}D^{\mu}$ controller [15], and the CRONE controller [16] are among the well-known fractional-order controllers. In those papers, it is verified that the fractional-order controllers are easier to implement and less cost compared to the traditional controllers.

Therefore, in this paper we present a new fractional-order controller, which only contains a single state, to synchronize the arbitrary 3D fractional-order chaos systems. The control approach which is based on the Lyapunov stability theory has the following three advantages. (1) It is so general that it can be applied to almost all 3D chaotic systems whether systems are incommensurate or commensurate. (2) It can synchronize the systems with little control cost and very fast. (3) It is very simple, easily realized experimentally, and more suitable for engineering applications. Numerical simulation results of synchronization of the fractional-order unified system, the fractional-order Liu system, and the fractional-order Chua–Hartley's system demonstrate the effectiveness and the validity of the proposed method.

## 2. Fractional derivatives and fractional dynamic systems

The fractional calculus plays an important role in modern science. In this paper we mainly use the Caputo fractional operators [1,2,17]. The Caputo definition of the fractional derivative, which sometimes called smooth fractional derivative, is described as

$$D_t^q = \begin{cases} \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t (t-\tau)^{m-q-1} f^{(m)}(t)d\tau, & m-1 < q < m, \\ \dfrac{d^m}{dt^m} f(t), & q = m, \end{cases} \quad (1)$$

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where $m$ is the first integer which is not less than $q$, and $\Gamma$ is the Gamma function,

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt. \tag{2}$$

When numerically solving the fractional differential equations, we also adopt the improved version of Adams–Bashforth–Moulton algorithm [2,18] which is proposed based on the predictor–correctors scheme. To explain the method we consider the following differential equation:

$$\begin{aligned} &D_t^q y(t)=r(t,y(t)), \quad 0\le t\le T,\\ &y^{(k)}(0)=y_0^{(k)}, \quad k=0,1,\cdots,m-1. \end{aligned} \tag{3}$$

The differential Eq. (3) is equivalent to Volterra integral equation in the following:

$$y(t)=\sum_{k=0}^{\lceil q\rceil-1} y_0^{(k)}\frac{t^k}{k!}+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}r(s,y(s))ds. \tag{4}$$

Now, set $h=T/N$, $t_n=nh(n=0,1,\ldots,N)$. The integral equation can be discretized as follows:

$$\begin{aligned} y_h(t_{n+1})=&\sum_{k=0}^{\lceil q\rceil-1} y_0^{(k)}\frac{t^k}{k!}+\frac{h^q}{\Gamma(q+2)}r(t_{n+1},y_h^p(t_{n+1}))\\ &+\frac{h^q}{\Gamma(q+2)}\sum_{j=0}^{n} a_{j,n+1}r(t_j,y_h(t_j)), \end{aligned} \tag{5}$$

where

$$y_h^p(t_{n+1})=\sum_{k=0}^{\lceil q\rceil-1} y_0^{(k)}\frac{t^k}{k!}+\frac{1}{\Gamma(q)}\sum_{j=0}^{n} b_{j,n+1}r(t_j,y_h(t_j)), \tag{6}$$

and

$$a_{j,n+1}=\begin{cases} n^{q+1}-(n-q)(n+1)^q, & j=0,\\ (n-j+2)^{q+1}+(n-j)^{q+1}-2(n-j+1)^{q+1}, & 1\le j\le n,\\ 1, & j=n+1, \end{cases} \tag{7}$$

$$b_{j,n+1}=\frac{h^q}{q}((n+1-j)^q-(n-j)^q). \tag{8}$$

The error of this approximation is described as follows:

$$\max_{j=0,1,\ldots,N}|y(t_j)-y_h(t_j)|=O(h^p). \tag{9}$$

where $p=\min(2, 1+q)$.

In this paper we mainly consider the order $0<q<1$. There are some general properties of the fractional-order derivative which are described as follows [1,2].

**Property 1.** *Caputo fractional derivative is a linear operator, i.e.,*

$$D_t^{\alpha}(\lambda f(t)+\mu g(t))=\lambda D_t^{\alpha}f(t)+\mu D_t^{\alpha}g(t), \tag{10}$$

*where $\lambda$, $\mu$ are real constants.*

**Property 2.** *Caputo fractional derivative satisfies additive index law(semigroup property)*

$$D_t^{\alpha}D_t^{\beta}f(t)=D_t^{\beta}D_t^{\alpha}f(t)=D_t^{\alpha+\beta}f(t), \tag{11}$$

*which holds under some reasonable constraints on the function $f(t)$.*

**Property 3.** *For the fractional-order nonlinear system $D_t^{\alpha}\boldsymbol{x}(t)=\boldsymbol{f}(\boldsymbol{x}(t))$, $\boldsymbol{f}(\boldsymbol{x}(t))$ satisfies the Lipschiz condition with respect to $\boldsymbol{x}$, i.e.,*

$$||\boldsymbol{f}(\boldsymbol{x}_1(t))-\boldsymbol{f}(\boldsymbol{x}_2(t))||\le l||\boldsymbol{x}_1(t)-\boldsymbol{x}_2(t)||, \tag{12}$$

*where $||\cdot||$ is 1-norm and $l$ is a positive constant. Especially, if $\boldsymbol{f}(\boldsymbol{x})=0$ at $\boldsymbol{x}=0$. It follows that*

$$||\boldsymbol{f}(\boldsymbol{x}(t))||\le l||\boldsymbol{x}(t)||. \tag{13}$$

In the following, we mainly consider a three-dimensional fractional-order nonlinear system

$$D_t^{\alpha}\boldsymbol{x}=\boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0)=\boldsymbol{c}, \tag{14}$$

where $\alpha(\alpha\in(0,1])$ is the fractional order of derivatives, $\boldsymbol{x}=(x_1(t), x_2(t), x_3(t))^T$ is the system state variable, $\boldsymbol{x}(0)=(c_1,c_2,c_3)^T$ is the initial value, and $D_t^{\alpha}$ denotes the Caputo fractional-order derivative operator [1]. The equilibrium points of system (14) are calculated via solving $\boldsymbol{f}(\boldsymbol{x}^*)=0$. Then we have the following conclusion for the stability of these equilibrium points.

**Theorem 1.** [2,19] *For $\alpha\in(0,1]$ the equilibrium point $\boldsymbol{x}^*$ of system (14) is globally asymptotically stable if all the eigenvalues $\lambda_i(i=1,2,3)$ of the Jacobian matrix $A=\partial\boldsymbol{f}/\partial\boldsymbol{x}$, evaluated at $\boldsymbol{x}^*$, satisfy the condition*

$$|arg(\lambda_i)|>\alpha\pi/2, \quad i=1,2,3. \tag{15}$$

## 3. Synchronization of the fractional-order chaotic system

For the three-dimensional fractional-order nonlinear system (14), we assume that:

$$\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{A}_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-\boldsymbol{x}), \tag{16}$$

where $\boldsymbol{A}_{\boldsymbol{x},\boldsymbol{y}}$ is a bounded matrix with its elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$. We consider the system (14) is the drive system, then the response system is given as

$$D_t^{\alpha}\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{y}). \tag{17}$$

We add a control function $\boldsymbol{\varphi}(\boldsymbol{u})$ to the system (17), then the controlled response system is

$$D_t^{\alpha}\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{y})+\boldsymbol{\varphi}(\boldsymbol{u}). \tag{18}$$

Let synchronization error $\boldsymbol{e}=\boldsymbol{y}-\boldsymbol{x}$, the error system from (14) and (18) is obtained

$$D_t^{\alpha}\boldsymbol{e}=\boldsymbol{f}(\boldsymbol{y})-\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{\varphi}(\boldsymbol{u})=\boldsymbol{A}_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{e}+\boldsymbol{\varphi}(\boldsymbol{u}). \tag{19}$$

The control function is usually given as

$$\boldsymbol{\varphi}(\boldsymbol{u})=-\boldsymbol{k}\boldsymbol{u}, \tag{20}$$

where $\boldsymbol{k}$ is the control parameter matrix which is always diagonal and $\boldsymbol{u}$ is defined as

$$D_t^{1-\alpha}\boldsymbol{u}=\boldsymbol{\omega}\boldsymbol{e}, \tag{21}$$

with $\boldsymbol{\omega}=\text{diag}(\omega_1,\omega_2,\omega_3)$ being a nonnegative diagonal matrix.

According to the Property 2 and formula (21), we have $\dot{u}=D_t^1\boldsymbol{u}=D_t^{\alpha}D_t^{1-\alpha}\boldsymbol{u}=\boldsymbol{\omega}D_t^{\alpha}\boldsymbol{e}$. And according to Properties 1 and 3 of the Caputo fractional derivative operator, there is a positive number $\lambda$ for Eq. (21), such that $||\boldsymbol{\omega}\boldsymbol{e}||\le\lambda||\boldsymbol{u}||$. Let $\omega^*$ be the minimum positive value of $\{\omega_1,\omega_2,\omega_3\}$, then $||\boldsymbol{e}||\le\lambda/\omega^*||\boldsymbol{u}||$ and if $\boldsymbol{u}=0$, then $\boldsymbol{e}=0$.

Then we can easily get that the systems (14) and (18) are synchronized if and only if the error system (19) is asymptotically stable at the origin point.

**Theorem 2.** *The fractional-order controller can control the error system (19) responses to the asymptotically stable at the origin point, i.e, systems (14) and (18) are globally asymptotically synchronized, if the*