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Torque equations for spin and orbital angular momenta of radiation fields and Faraday effect

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ABSTRACT

The spin angular momentum $\boldsymbol{S}$ of light has never been linked to the Faraday rotation of light traveling in an optically active medium possessing a rotational invariance of a crystal, because there was no helicity term associated with the phase shift in the previous torque equation for $\boldsymbol{S}$. In order to relate the change in $\boldsymbol{S}$ with time to the Faraday rotation, therefore, we derived an exact torque equation for $\boldsymbol{S}$. As a result, a magnetic helicity term appeared in a new torque equation for $\boldsymbol{S}$, so that one-half of the phase shift derived from the helicity term was equivalent to the Faraday rotation angle. However, the orbital angular momentum $\boldsymbol{L}$ had no relation to the Faraday rotation. It was thus clarified that the change in $\boldsymbol{S}$ with time is related to the Faraday rotation angle of light traveling in an optically active medium, owing to the appearance of the helicity term without a rotational invariance around the optical axis. It was also demonstrated theoretically that the Faraday rotation is accompanied by a torque acting on the crystal so that the total angular momentum of light and matter is conserved.

1. Introduction

The angular momentum carried by light can be characterized by the spin angular momentum associated with circular polarization and the orbital angular momentum associated with the spatial distribution of the wave. Both spin and orbital angular momenta of a light beam have in fact been measured [1–4]. Theoretically, the spin and orbital angular momenta of the radiation fields in free space have been defined as $\boldsymbol{S}=\int_\tau(\boldsymbol{D}\times\boldsymbol{A})\mathrm{d}\nu$ and $\boldsymbol{L}=\int_\tau\sum_{i=x,y}D_i(\boldsymbol{r}\times\nabla)A_i\mathrm{d}\nu$, where the electric induction field $\boldsymbol{D}$ is related to the electric field $\boldsymbol{E}$ as $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}$, where ε_0 is the permittivity of free space and τ is an arbitrary volume [5–8]. The spin angular momentum arises even in the transverse plane electromagnetic waves. However, the orbital angular momentum never appears in the plane electromagnetic waves but arises sometimes when the phase gradient satisfies certain conditions providing the transverse energy circulation within the beam [9,10].

The linearly polarized light entering the photorefractive crystal is decomposed into its constituents, which travel in a medium with different velocities. This will result in a phase difference θ between the two (right and left) circularly polarized waves. After coming out from the crystal, these waves will be recombined to a linearly polarized light with the plane of polarization rotated by θ_F from the plane of polarization of an incident polarized one. This phenomenon is referred to as the Faraday effect. The Faraday rotation has hitherto been explained well by the phenomenological theory [11], but it has never been connected with the change in $\boldsymbol{S}$ of light traveling in an optically active medium. One of the reasons is that the helicity term associated with the phase shift of the electromagnetic waves was neglected in the previous torque equation for $\boldsymbol{S}$ so that no torque for $\boldsymbol{S}$ was induced in an optically active medium [5–7]. For this reason, we made an attempt to derive an exact torque equation of $\boldsymbol{S}$ so that the change in $\boldsymbol{S}$ with time in such a medium is connected with the Faraday rotation.

The purpose of this study is to derive the correct torque equations for $\boldsymbol{S}$ and $\boldsymbol{L}$ so that the change in $\boldsymbol{S}$ with time of light propagating in an optically active medium is related to the Faraday rotation, and is to examine how the total angular momentum of the field and crystal is conserved in such a medium.

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2. The relation between angular momentum and crystal symmetry

The Maxwell equations for the macroscopic electromagnetic fields in a medium are given by [12]

$$\nabla \times \boldsymbol{E} = \frac{\partial \boldsymbol{B}}{\partial t} \tag{1}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{I} \tag{2}$$

$$\nabla \boldsymbol{D} = \rho \tag{3}$$

$$\nabla \boldsymbol{B} = 0 \tag{4}$$

It is assumed here that the medium is inhomogeneous and anisotropic but is free of dispersion. In these equations, the electric $\boldsymbol{D}$ and magnetic $\boldsymbol{B}$ induction fields are related to the field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ as $D_{i=x,y,z} = \Sigma_{j=x,y,z}\varepsilon_{ij}E_j$ and $B_{i=x,y,z} = \Sigma_{j=x,y,z}\mu_{ij}H_j$ where ε_{ij} and μ_{ij} represent the permittivity and permeability tensors, $\boldsymbol{I}$ is the electric current density, and ρ is the charge density.

Let us consider an arbitrary volume τ inside the crystal filled by charges, described by the volume density ρ and current density $\boldsymbol{I}$. Because of the interaction with the electromagnetic field, the charges experience a total mechanical torque $\mathrm{d}\boldsymbol{L}_c/\mathrm{d}t$ given by [12]

$$\frac{\mathrm{d}\boldsymbol{L}_c}{\mathrm{d}t} = \int_\tau \mathrm{d}v \boldsymbol{r} \times [\rho\boldsymbol{E} + (\boldsymbol{I} \times \boldsymbol{B})] \tag{5}$$

where $\boldsymbol{L}_c$ is the mechanical angular momentum of the (charged and neutral) particles. Substituting $\boldsymbol{I}$ and ρ from Eqs. (2) and (3) into Eq. (5), we get, after some manipulations,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{L}_c + \boldsymbol{J}) = \int_\tau \mathrm{d}v \left\{\boldsymbol{r} \times [\boldsymbol{E}(\nabla \boldsymbol{D}) - \boldsymbol{D} \times (\nabla \times \boldsymbol{E}) - \boldsymbol{B} \times (\nabla \times \boldsymbol{H})]\right\} \tag{6}$$

where $\boldsymbol{J}$ is the angular momentum of a spatial distribution of electromagnetic fields in a medium and expressed as

$$\boldsymbol{J} = \int_\tau \mathrm{d}v [\boldsymbol{r} \times (\boldsymbol{D} \times \boldsymbol{B})] \tag{7}$$

in analogy to the interpretation of the linear momentum density $\boldsymbol{D} \times \boldsymbol{B}$ in a medium [6,12]. This definition is valid at least when the medium is linear, but not necessarily isotropic, in its response [12]. However, we do not enter here into the well-known question of the correct definition of momentum in media [12,13].

The resultant torque equation of $\boldsymbol{L}_c$ and $\boldsymbol{J}$ is expressed using the volume integral as [12]

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{L}_c + \boldsymbol{J}) = \int_\tau \mathrm{d}v (\boldsymbol{r} \times \nabla T) \tag{8}$$

where ∇T is the well-known Maxwell stress tensor, given by

$$\nabla T = \boldsymbol{E}(\nabla \boldsymbol{D}) - \boldsymbol{D} \times (\nabla \times \boldsymbol{E}) - \boldsymbol{B} \times (\nabla \times \boldsymbol{H}) \tag{9}$$

When the crystal shows a rotational symmetry around the optical axis, however, we can give a correct definition of the angular momentum. The reason is that the angular momentum of the crystal along the optical axis is a quantity not mixing with the angular momentum of the electromagnetic field; the rotational invaiance around the optical axis decouples the contribution of the cyrstal and that of the field [14].

The vector $(\boldsymbol{r} \times \nabla T)$ in the volume integral of Eq. (8) can be rewritten as

$$\boldsymbol{r} \times \nabla T = \nabla F + \boldsymbol{g} \tag{10}$$

where the tensor F and the vector $\boldsymbol{g}$ are defined by

$$F_{ij} = \varepsilon_{mn} x_m T_{nj} \tag{11}$$

$$g_i = \varepsilon_{mn} T_{mn} \tag{12}$$

and ε_{imn} is the completely antisymmetric tensor of rank 3 (Levi-Cività symbol) [14]. The Maxwell stress tensor T_{ij} is expressed as

$$T_{ij} = E_i D_j + H_i B_j - \frac{\delta_{ij}}{2}(E_m D_m + H_m B_m) \tag{13}$$

where the indices i and j denote x, y and z and δ_{ij} is the Kronecker delta function. Substituting Eq. (10) into the right hand side of Eq. (8), it is transformed into a surface integral and a volume integral as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{L}_c + \boldsymbol{J}) = \int_\Sigma (F\boldsymbol{n})\mathrm{d}S + \int_\tau \boldsymbol{g}\mathrm{d}v \tag{14}$$

where $\boldsymbol{n} = n_x\boldsymbol{e}_1 + n_y\boldsymbol{e}_2 + n_z\boldsymbol{e}_3$ is a unit vector pointing outward from the surface. Eq. (14) can be interpreted as the balance equation for the angular momentum only if the volume integral of the vector $\boldsymbol{g}$ vanishes [14]. The explicit expression of $\boldsymbol{g}$ is obtained from Eqs. (12) and (13) as

$$\boldsymbol{g} = (T_{yz} - T_{zy})\boldsymbol{e}_1 + (T_{zx} - T_{xz})\boldsymbol{e}_2 + (T_{xy} - T_{yx})\boldsymbol{e}_3 \tag{15}$$

When two monochromatic plane waves propagate along the z axis in an optically active medium, the z component of $\boldsymbol{g}$ vanishes because of $T_{xy} = T_{yx}$ as a consequence of the rotational invariance around the z axis of the crystal. Therefore, the z component of the right hand side of Eq. (14) can be expressed by only the surface integral over the boundary Σ enclosing the volume τ as

$$\frac{\partial}{\partial t}(L_{cz} + J_z) = \int_\tau \mathrm{d}S(F_{zx}n_x + F_{zy}n_y + F_{zz}n_z) \tag{16}$$

where $F_{zi} = xT_{yi} - yT_{xi}$ for $i = x$, y and z. Thus, this expression allows us to interpret J_z as the z component of the angular momentum of the radiation stored in the volume τ and the surface integral as the flux of angular momentum incoming through the boundary Σ [14]. When a medium possesses the rotational invariance around the z axis and the fields are described by the plane waves, however, the surface integrals disappear and Eq. (16) is expressed simply as

$$\frac{\partial}{\partial t}(L_{cz} + J_z) = 0 \tag{17}$$

so that the total angular momentum of $(L_{cz} + J_z)$ is conserved in a medium. It makes it possible to distinguish between $\partial L_{cz}/\partial t$ and $\partial J_z/\partial t$, as will be mentioned later.

3. Torque equations for spin and orbital angular momenta of radiation fields

The electromagnetic field can be separated into transverse and longitudinal fields, which have by definition a vanishing divergence and curl, respectively. The magnetic field is purely transverse, while the longitudinal electric induction field $\boldsymbol{D}_\parallel$ is given by the instantaneous Coulomb field arising from the charge density ρ, i.e., $\nabla \boldsymbol{D}_\parallel = \rho$. The transverse electric induction field $\boldsymbol{D}_\perp$ thus describes the radiation part, which contains in fact the only real dynamical degrees of freedom of the field. For $\boldsymbol{D}_\perp$ alone, therefore, Eq. (3) may be rewritten as

$$\nabla \boldsymbol{D}_\perp = 0 \tag{18}$$

The electric field $\boldsymbol{E}$ and magnetic induction field $\boldsymbol{B}$ are then expressed in the radiation gauge by the vector potential $\boldsymbol{A}$ defined as [12]

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} \tag{19}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{20}$$

where we used the radiation gauge, in which the scalar potential vanishes in the absence of charges; the vector potential $\boldsymbol{A}$ is taken