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Generalized Gabor expansion associated with linear canonical transform series

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ABSTRACT

The Gabor expansion (GE), which maps the time domain signal into the joint time and frequency domain, has been recognized as very useful for signal processing. However, sinusoidal analysis used in the traditional GE is not appropriate for a compact representation for chirp-type signals. In this paper, a generalized Gabor expansion (GGE) is proposed in order to rectify the limitations of the GE, the proposed expansion not only inherits the advantage of GE, but also has the capability of signal representations in the linear canonical transform (LCT) domain which is similar to the LCT. Basis functions of the proposed expansion are obtained via LCT basis. Compared with the traditional GE, the GGE can offer signal representations on a general, non-rectangular time–frequency plane tiling. Besides, the completeness and biorthogonality conditions of the GGE are derived.

1. Introduction

Time–frequency (TF) analysis provides a characterization of signals in terms of joint time and frequency content [1,2]. One of the fundamental issues in the TF analysis is obtaining of distribution of signal energy over joint TF plane with sufficient time and frequency resolutions. The Gabor expansion (GE) has been shown to be an appropriate tool for TF analysis [3–5]. It represents a signal in terms of time and frequency shifted basis functions, and has been used in various applications to analyze the time-varying frequency content of signal [6,7]. However, under the extension of research objects and scope, the GE has been discovered to have short-comings. Since basis functions of the Gabor representation are obtained by translating and modulating with sinusoids a signal window function, resulting in a fixed and rectangular TF sampling lattice. Therefore, sinusoidal analysis used in the traditional GE is not appropriate for a compact representation for chirp-type signals, which are ubiquitous in nature and man-made systems, are this kind of signals.

In order to rectify the limitations of GE, many of approaches have been proposed to improve the solution of Gabor representations [6,8–17]. Such as: using large dictionary of basis functions [6,10], averaging results obtained using different windows [9], maximizing energy concentration measures [8,11,12] and using signal-adaptive basis functions to match the time-varying frequency of the signal [13]. In recent works, the fractional Gabor expansion representations on a general, non-rectangular TF lattice have attracted a considerable attention [14–17]. A non-rectangular lattice is more appropriate for the TF analysis of signals with time-varying frequency content. Simultaneously, comparing to the FRFT with one extra degree of freedom, linear canonical transform (LCT) is more flexible for its extra three degree of freedom, and has been used frequently in time–frequency analysis and non-stationary signal processing (particular for chirp signal) [18–26]. Inspired of fractional GE, we introduce the concept of the generalized Gabor expansion (GGE), combining the idea of GE and LCT. The proposed transform not only inherits the advantages of GE, but also has the capability of signal representations in the LCT domain which is similar to LCT. In this paper, we present GGE for the time–frequency representation of chirp signals. The new representation tiles the TF plane in parallelogram shapes which clearly is a better way of representing chirp signals than the traditional rectangular grid. The basis functions of the proposed expansion are obtained via LCT basis and they are impulses in the LCT domain. As a result, the time-varying frequency content of a signal is represented better than with sinusoidal modulated expansions. Besides, the biorthogonality relation between the synthesis and analysis functions for the proposed expansion is derived.

The rest of this paper is organized as follows. Section 2 presents the theoretical basis of GE, LCT and linear canonical series. In Section 3, the GGE is proposed, moreover, the completeness and

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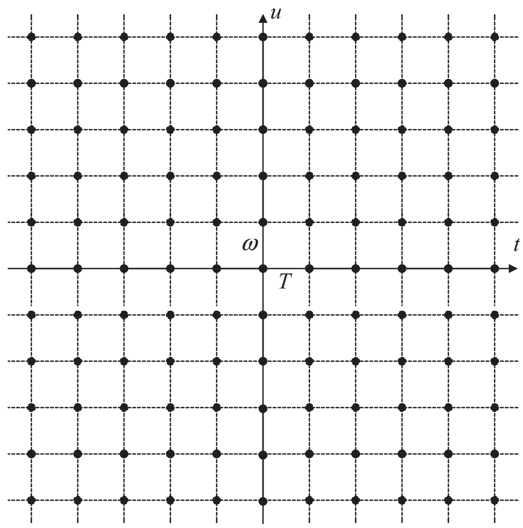

Fig. 1. Rectangular time–frequency plane sampling lattice used in the Gabor expansion.

biorthogonality conditions of the GGE are derived. Finally, Section 4 concludes this paper.

2. Preliminaries

In this section, we give brief background on the Gabor expansion (GE) and linear canonical transform (LCT). Also, we give an introduction to the linear canonical series (LCS) expansion.

2.1. The Gabor expansion

The traditional GE represents a signal as s combination of time and frequency translated basis functions, and has been used in various applications to analyze the time-varying frequency content of a signal. For signal $f(t)$, the GE is defined as

$$f(t)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{m,n}h_{m,n}(t) \tag{1}$$

$$h_{m,n}(t)=h(t-mT)\,e^{jn\omega t} \tag{2}$$

where T and ω represent time and frequency sampling intervals, respectively. The synthesis function $h(t)$ is subject to a unit energy constraint.

$$\int_{-\infty}^{+\infty}\left|h(t)\right|^2 dt=1 \tag{3}$$

Once a specific window type is selected, there remain two free parameters T and ω, whose choice is crucial as it directly effects the existence, uniqueness, convergence and numerical stability of the expansion. The conventional constrains,$\omega T=2\pi$, advocated by Gabor are sensible and indeed optimal by criteria such as minimum sampling rate and numerical stability. $\omega T<2\pi$ is called over-sampling which results in redundancy in the Gabor coefficients, and $\omega T>2\pi$ is called under-sampling which causes a loss of information [4]. The GE represents a signal in terms of time and frequency shifted basis functions, called TF atoms. This type of basis functions generates a fixed and rectangular TF plane sampling. An example of such a sampling geometry is shown in Fig. 1.

Although the GE has been recognized as a very useful tool in signal processing, its applications were limited due to the difficulties associated with computing the Gabor coefficients $a_{m,n}$. In general, the set of time and frequency shifted window functions $h_{m,n}$ do not form an orthogonal basis for the square-summable continuous functions space $L_2(R)$.

One solution to this problem, developed by Bastianns [14], is to introduce an auxiliary function $\gamma(t)$ called the biorthogonal function. Then the Gabor coefficients $\{a_{m,n}\}$ are evaluated via the use of the so called biorthogonal function, defined via

$$a_{m,n}=\int_{-\infty}^{+\infty} f(t)\gamma^*_{m,n}(t)\,dt \tag{4}$$

where

$$\gamma_{m,n}(t)=\gamma(t-mT)\,e^{jn\omega t} \tag{5}$$

and the asterisk denote complex conjugation. $\gamma(t)$ in (5) can also be considered as an analysis function. Substituting (4) into (1) leads to the completeness relation

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} h_{m,n}(t)\gamma^*_{m,n}(t')=\delta(t-t') \tag{6}$$

where $\delta(\cdot)$ denotes the Dirac δ-function. Applied to the n-summation of the Poisson-sum formula [27], the completeness relation (6) leads to the following biorthogonality relationship between $\gamma(t)$ and $h(t)$:

$$\frac{2\pi}{\omega}\sum_{m=-\infty}^{\infty} h(t-mT)\gamma^*\left(t-\left[m+n\frac{2\pi}{\omega T}\right]\right)=\delta_n \tag{7}$$

where $n=0,\pm1,\pm2,\ldots$, and the factor $2\pi/\omega T$ represents the degree of over-sampling. δ_j indicates a Kronecher delta [4].

2.2. Linear canonical transform and linear canonical series

The linear canonical transform (LCT) provides a mathematical model of paraxial propagation though quadratic phase systems [19]. The LCT, which is a generalization of the FT, generalizes the usual time and frequency domain representations of the signals to the continuum of infinite LCT domains. The LCT of a signal $f(t)$ with parameter $A=(a, b; c, d)$ is defined as [18–20]:

$$F^A(u)=L^A[f(t)](u)=\begin{cases}\int_{-\infty}^{\infty} f(t)K_A(t,u)\,dt, & b\neq 0,\\ \sqrt{d}e^{j1/2cdu^2}f(du), & b=0.\end{cases} \tag{8}$$

where

$$K_A(t,u)=\sqrt{\frac{1}{j2\pi b}}\,e^{j1/2\left[(a/b)t^2-(2/b)tu+(d/b)u^2\right]} \tag{9}$$

a, b, c, d are real numbers satisfying $ad-bc=1$. It should be noted that, when $b=0$, the LCT of a signal is essentially a chirp multiplication and it will not be discussed in this paper.

The transform matrix A is useful in the analysis of optical systems because if several systems are cascaded, the overall system matrix can be found by multiplying the corresponding matrices. The LCT family includes the Fourier and fractional Fourier transforms, coordinate scaling, and chirp multiplication and convolution operations as its special cases [19].

In the following, based on the LCT, the linear canonical series (LCS) is introduced, which is the generalized pattern of Fourier series [28,29]. In can reveal the mixed time and frequency