Optik 125 (2014) 4491–4496

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.de/ijleo

Generalized wavelet transform based on the convolution operator in the linear canonical transform domain

Deyun Wei*, Yuan-Min Li

Department of Mathematics, Xidian University, Xi'an 710071, China

ARTICLE INFO

Article history:
Received 7 August 2013
Accepted 12 February 2014

Keywords:
Wavelet transform
Time-frequency analysis
Linear canonical transform

ABSTRACT

The wavelet transform (WT) and linear canonical transform (LCT) have been shown to be powerful tool for optics and signal processing. In this paper, firstly, we introduce a novel time-frequency transformation tool coined the generalized wavelet transform (GWT), based on the idea of the LCT and WT. Then, we derive some fundamental results of this transform, including its basis properties, inner product theorem and convolution theorem, inverse formula and admissibility condition. Further, we also discuss the time-fractional-frequency resolution of the GWT. The GWT is capable of representing signals in the time-fractional-frequency plane. Last, some potential applications of the GWT are also presented to show the advantage of the theory. The GWT can circumvent the limitations of the WT and the LCT.

1. Introduction

The wavelet transform (WT), which has had a growing importance in optics and signal processing, has been shown to be a successful tool for time-frequency analysis and image processing [1]. It has found many applications in time-dependent frequency analysis of short-transient signals, data compression, optical correlators, sound analysis, representation of fractal aggregates and many others [1–7]. However, the signal analysis capability of the WT is limited in the time-frequency plane. Therefore, the WT is inefficient for processing signals whose energy is not well concentrated in the frequency domain.

Now, many of novel signal processing tools have been proposed to rectify the limitations of the WT, and can provide signal representation in the fractional domain. Such as fractional Fourier transform (FRFT) [8], the Radon–Wigner transform [9], the fractional wavelet transform (FRWT) [10–14], fractional wave packet transform (FRWPT) [15], the short-time FRFT [16], the LCT [17–20] and so on. In the past decade, although the FRFT has attracted much attention of the signal processing community, it cannot obtain information about local properties of the signal. Therefore, the FRWT fails in obtaining information about local properties of the signal [10–12]. Simultaneously, the drawback of the short-time FRFT is that its time and fractional domain resolutions cannot simultaneously be arbitrarily high [16]. The FRWPT did not receive much attention for the lack of physical interpretation and high computation complexity [15]. The LCT [17–20], which was introduced during the 1970s with four parameters, has been proven to be one of the most powerful tools for non-stationary signal processing. The well-known signal processing operations, such as the Fourier transform (FT), the FRFT, the Fresnel transform, and the scaling operations are all special cases of the LCT [18,19]. The LCT has also found many applications in the solution of optical systems, filter design, time-frequency analysis and many others [21–31]. This transform, however, has one major drawback due to using global kernel, i.e., the LCT representation only provides such LCT spectral content with no indication about the time localization of the LCT spectral components [19,20]. Therefore, the analysis of non-stationary signals whose LCT spectral characteristics change with time requires joint signal representations in both time and LCT domains, rather than just a LCT domain representation.

As a generalization of the WT, a novel FRWT can combine the advantages of the WT and the FRFT, i.e., it is a linear transformation without cross-term interference and is capable of providing multiresolution analysis and representing signal in the fractional domain [13,14]. Simultaneously, comparing to the FRFT with one extra degree of freedom, LCT is more flexible for its extra three degree of freedom, and has been used frequently in time-frequency analysis and non-stationary signal processing. Inspired of FRWT, we introduce the concept of the generalized wavelet transform (GWT), combining the idea of LCT and WT. the proposed transform not only inherits the advantages of multiresolution analysis of the WT, but also has the capability of signal representations in the LCT domain which is similar to LCT. Compared with the existing FRWT, the GWT can offer signal representations in the time-fractional-frequency

* Corresponding author.
E-mail addresses: deyunwei@yahoo.com.cn, dywei@xidian.edu.cn (D. Wei), yuanminlihit@yahoo.com.cn, ymli@xidian.edu.cn (Y.-M. Li).

http://dx.doi.org/10.1016/j.ijleo.2014.02.021

plane in LCT domain. Besides, it has explicit physical interpretation, low computation complexity and usefulness for practical application.

The rest of this paper is organized as follows. Section 2 presents the theoretical basis of WT, LCT and convolution theory. In Section 3, the GWT is proposed. Moreover, some fundamental results of this transform are presented, including its basis properties, theorems, inverse formula and admissibility condition. In Section 4, the time-fractional-frequency analysis of the GWT is discussed. Potential applications for GWT are presented in section 5. Finally, Section 6 concludes this paper.

2. Preliminaries

2.1. Fourier transform and wavelet transform

Fourier transform (FT) is a tool widely applied for signal processing. In this paper, the FT is defined as follows [28]:

$$F(u) = \mathscr{F}(f(x))(u) = \int_{-\infty}^{\infty} f(x)\, e^{-jux}\, dx \tag{1}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{jux}\, du \tag{2}$$

where $\mathscr{F}$ denotes the FT operator and is also used in the subsequent sections. In the following section, Asymmetric definition of the FT is utilized in this paper.

The conventional convolution of two signals $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)\, d\tau = \left\langle f(.), \bar{g}(x-.)\right\rangle \tag{3}$$

where * and the bar-in the subscript denote the conventional convolution operator and the complex conjugate, respectively, and $\langle ., .\rangle$ indicates the inner product. To be specific, the convolution theorem of the FT for the signals $f(x)$ and $g(x)$ with associated FTs, $F(u)$ and $G(u)$, respectively is given by:

$$f(x) * g(x) \overset{\mathscr{F}}{\longleftrightarrow} F(u)G(u) \tag{4}$$

A one-dimensional wavelet transform (WT) of a signal $f(x)$ is defined as [1]

$$W_f(a,b) = W(f(x))(u) = \int_{-\infty}^{\infty} f(x)\, \bar{h}_{ab}(x)\, dx \tag{5}$$

It can be also defined as a conventional convolution, i.e.,

$$W_f(a,b) = f(x) * (a^{-1/2}\bar{h}(-x/a)) = \left\langle f(.), h_{ab}(.)\right\rangle \tag{6}$$

where the kernel $h_{ab}(x)$ is a continuous affine transformation of the mother wavelet function $h(x)$,

$$h_{ab}(x) = \frac{1}{\sqrt{a}} h\left(\frac{x-b}{a}\right) \tag{7}$$

where b is the shift amount, a is the scale parameter, and $\sqrt{a}$ is the normalization factor. Based on the conventional convolution theorem and inverse FT, the WT of the signal $f(x)$ can be also expressed as:

$$W_f(a,b) = \int_{-\infty}^{\infty} \sqrt{a}F(u)\, \bar{H}(au)\, e^{jub}\, du \tag{8}$$

where $F(u)$ and $H(u)$ denote the FT of $f(x)$ and $h(x)$, respectively. Since $H(0) = \int_{-\infty}^{+\infty} h(x)\, dx = 0$, each wavelet component is actually a differently scaled bandpass filter, the wavelet transform is a localized transformation and thus is efficient for processing transient signals. However, WT is inefficient for processing signals whose energy is not well concentrated in the frequency domain. Thus, signal analysis associated with it is limited to the time-frequency plane.

2.2. Linear canonical transform and convolution theorem

The linear canonical transform (LCT) provides a mathematical model of paraxial propagation though quadratic phase systems. The output light field $F_T(u)$ a quadratic phase systems is related to its input field $f(x)$ through [19]

$$F_T(u) = L^T[f(x)](u) = \begin{cases} \int_{-\infty}^{\infty} f(x)K_T(u,x)\, dx, & b \neq 0, \\ \sqrt{D}e^{j(1/2)CDu^2} f(Du), & b = 0, \end{cases} \tag{9}$$

where

$$K_T(u,x) = \sqrt{\frac{1}{(j2\pi B)}}\, e^{j(1/2)\left[(A/B)x^2-(2/B)xu+(D/B)u^2\right]}, \tag{10}$$

where L^T is the unitary LCT operator with parameter matrix $T = (A, B; C, D)$, A,B,C,D are real numbers satisfying $AD-BC=1$. The inverse transform for LCT is given by a LCT having parameter $T^{-1} = (D, -B; -C, A)$, that is

$$f(x) = \int_{-\infty}^{\infty} F_T(u)\bar{K}_T(u,x)\, du \tag{11}$$

The transform matrix T is useful in the analysis of optical systems because if several systems are cascaded, the overall system matrix can be found by multiplying the corresponding matrices. It should be noted that, when $B=0$, the LCT of a signal is essentially a chirp multiplication and it is of no particular interest to our objective in this work, so it will not be discussed in this paper. The LCT family includes the FT and FRFT, coordinate scaling, and chirp multiplication and convolution operations as its special cases. For further details about the definition and properties of LCT, [17–20] can be referred.

Contrast with FT, the LCT has a number of unique properties, and it has been widely applied in optics and signal processing. However, the LCT is a global transformation, it cannot obtain information about local properties of the signal. In other word, the LCT tells us the fractional frequencies that exist across the whole duration of the signal but not the fractional frequencies which exist only at a particular time.

A convolution and product structures of the LCT is introduced in [28]

$$f(x) \odot g(x) = e^{-jAx^2/(2B)}\left[(f(x)\, e^{jAx^2/(2B)}) * g(x)\right] \tag{12}$$

where $\odot$ is the generalized convolution operation for the LCT. Then, the convolution theorem of the LCT for the signal $f(x)$ and $g(x)$is given by

$$f(x) \odot g(x) \overset{L^T}{\longleftrightarrow} F_T(u)G(u/B) \tag{13}$$

where $F_T(u)$ and $G(u)$ denotes the LCT of $f(x)$ and the FT of $g(x)$, respectively. Particularly, when $T=(0, 1; -1, 0)$, (12) reduces to the conventional convolution as given by (3).

3. Generalized wavelet transform

3.1. Definition of generalized wavelet transform

In this subsection, we defined a generalized wavelet transform (GWT) based on the convolution operation in LCT domain. The GWT of a square integrable signal $f(x)$ is defined as