Optik 124 (2013) 4425–4428

Contents lists available at ScienceDirect

Optik

journal homepage: www.elsevier.de/ijleo

Rigorous formula for electromagnetic field power based on Poynting vector

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ARTICLE INFO

Article history:
Received 23 August 2012
Accepted 10 January 2013

PACS:
03.50.De
42.30.Kq

Keywords:
Electromagnetic power
Angular spectrum
Poynting vector

ABSTRACT

Power of an electromagnetic field in a plane is derived exactly as a function of the angular spectrum of only electric field components at another plane based on Maxwell equations and Poynting vector. Then, this quantity is acquired as a function of the electric field components. In this calculation, a function appears that is general and does not depend on the electromagnetic field function.

1. Introduction

In optics the directly measurable quantity is optical power. For a plane wave, the two electric and magnetic fields are perpendicular to each other and their magnitudes hold in $E = cB$. Hence, this quantity is proportional to the squared magnitude of electric field amplitude [1,2]. In many of optical experiments these circumstances almost hold, and the intensity and total power are derived correctly by this process.

However, for a general electromagnetic field two electric and magnetic fields generally are not perpendicular to each other and their magnitudes don't hold in $E = cB$. In these cases, the common method is finding the magnetic field and then building corresponding Poynting vector. Then, by using this quantity, intensity and optical power is obtained.

On the other hand, energy balancing issue is a basic and challenging one in every field of physics. In optics, verification of conventional interpretation of Poynting vector, as the power flux, is an attractive issue [3].

In this paper the optical power transmitting through a plane is derived exactly as a function of the electric field specified at that plane or any plane parallel to it. In Section 2 the magnetic field is specified as a function of the angular spectrum of the electric field at a plane. Transmitted power based on Poynting vector is derived as a function of the angular spectrum of the electric field, and also as a function of the electric field, in Section 3.

2. Electric and magnetic fields

Suppose that the complex amplitude of vectorial electric field at xy plane is $\mathbf{E}(x, y, 0)$. The complex amplitude of electric field $\mathbf{E}(x, y, z)$ at plane $z > 0$ is [1,2]

$$E_\perp(x,y,z) = \int\int_{-\infty}^{\infty} A_\perp\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\exp\left[j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y+\frac{\gamma}{\lambda}z\right)\right]d\left(\frac{\alpha}{\lambda}\right)d\left(\frac{\beta}{\lambda}\right) \tag{1}$$

$$(\perp = x, y), \tag{1}$$

$$E_z(x,y,z) = \int\int_{-\infty}^{\infty} A_z\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\exp\left[j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y+\frac{\gamma}{\lambda}z\right)\right]d\left(\frac{\alpha}{\lambda}\right)d\left(\frac{\beta}{\lambda}\right), \tag{2}$$

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http://dx.doi.org/10.1016/j.ijleo.2013.01.064

where λ is the wavelength, $A_i(\alpha/\lambda, \beta/\lambda)$ is the Fourier transform of $E_i(x, y, 0)$, and

$$\gamma = (1-\alpha^2-\beta^2)^{1/2}, \qquad \alpha^2+\beta^2 \le 1, \tag{3}$$

$$\gamma = j(\alpha^2+\beta^2-1)^{1/2}, \qquad \alpha^2+\beta^2 > 1. \tag{3}$$

Because $\nabla . \mathbf{E}(x, y, z) = 0$,

$$A_z = -\left(\frac{\alpha}{\gamma}A_x + \frac{\beta}{\gamma}A_y\right). \tag{4}$$

The above three equations can be compressed to one vectorial equation:

$$\mathbf{E}(x,y,z) = \int\int_{-\infty}^{\infty} \mathbf{A}\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) \exp\left[j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y+\frac{\gamma}{\lambda}z\right)\right] d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right). \tag{5}$$

The second Maxwell equation (Faraday law) implies [4]

$$\nabla\times\mathbf{E} = jkc\mathbf{B}, \tag{6}$$

where $k = 2\pi/\lambda$ and c is velocity of light in the medium. Combination of Eqs. (5) and (6) gives

$$\mathbf{B}(x,y,z) = \int\int_{-\infty}^{\infty} \left[-\frac{1}{c}\mathbf{A}\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\times(\alpha\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}+\gamma\hat{\mathbf{z}})\right] \times \exp\left[j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y+\frac{\gamma}{\lambda}z\right)\right] d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right). \tag{7}$$

Interpretation of the above equation is simple. Total **B** is vectorial sum of several (infinite) vectors. Each vector is perpendicular to the electric field and the propagation direction (for real γ) of a plane wave and its magnitude is equal to $A(\alpha/\lambda, \beta/\lambda)d(\alpha/\lambda)d(\beta/\lambda)/c$. Thus, it is a magnetic field corresponding to $\mathbf{A}(\alpha/\lambda, \beta/\lambda)d(\alpha/\lambda)d(\beta/\lambda)/c$.

3. Transmitted power based on Poynting vector

In term of complex amplitude the flow of electromagnetic power, **S**, is governed by [5]

$$\mathbf{S} = \frac{1}{2}\mathrm{Re}\{\mathbf{E}\times \mathrm{H}*\}. \tag{8}$$

Now we examine the transmitted power, P, through the z plane.

$$P = \int\int_{-\infty}^{\infty} \mathbf{S}.\hat{\mathbf{z}}dxdy = \frac{1}{2\mu}Re\{\int\int_{-\infty}^{\infty} E_xB_y^*dxdy - \int\int_{-\infty}^{\infty} B_x^*E_ydxdy\}, \tag{9}$$

where μ is the magnetic permeability of the medium. By using Eqs. (1) and (7) the first integral in right hand side of (9) is

$$\int\int_{-\infty}^{\infty} dxdy \int\int_{-\infty}^{\infty} A_x\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) \exp\left[j2\pi\left(\frac{\alpha}{\lambda}x+\frac{\beta}{\lambda}y+\frac{\gamma}{\lambda}z\right)\right] d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right) \times \int\int_{-\infty}^{\infty} \frac{1}{c}\left[\gamma' * A_x^*\left(\frac{\alpha'}{\lambda},\frac{\beta'}{\lambda}\right) - \alpha' A_z^*\left(\frac{\alpha'}{\lambda},\frac{\beta'}{\lambda}\right)\right]$$
$$\times \exp\left[-j2\pi\left(\frac{\alpha'}{\lambda}x+\frac{\beta'}{\lambda}y+\frac{\gamma'*}{\lambda}z\right)\right] d\left(\frac{\alpha'}{\lambda}\right) d\left(\frac{\beta'}{\lambda}\right). \tag{10}$$

By interchanging the order of integrals

$$\int\int_{-\infty}^{\infty} E_xB_y^*dxdy = \int\int_{-\infty}^{\infty} A_x\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right) \times \int\int_{-\infty}^{\infty} \frac{1}{c}\left[\gamma' * A_x^*\left(\frac{\alpha'}{\lambda},\frac{\beta'}{\lambda}\right) - \alpha' A_z^*\left(\frac{\alpha'}{\lambda},\frac{\beta'}{\lambda}\right)\right]$$
$$\times \exp\left[j\frac{2\pi z}{\lambda}(\gamma-\gamma'*)\right] d\left(\frac{\alpha'}{\lambda}\right) d\left(\frac{\beta'}{\lambda}\right) \times \int\int_{-\infty}^{\infty} \exp\left[j2\pi\left(\frac{\alpha-\alpha'}{\lambda}x+\frac{\beta-\beta'}{\lambda}y\right)\right] dxdy. \tag{11}$$

But

$$\int\int_{-\infty}^{\infty} \exp\left[j2\pi\left(\frac{\alpha-\alpha'}{\lambda}x+\frac{\beta-\beta'}{\lambda}y\right)\right] dxdy = \delta\left(\frac{\alpha-\alpha'}{\lambda},\frac{\beta-\beta'}{\lambda}\right), \tag{12}$$

therefore,

$$\int\int_{-\infty}^{\infty} E_xB_y^*dxdy = \frac{1}{c}\int\int_{-\infty}^{\infty} A_x\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) \times \left[\gamma * A_x^*\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right) - \alpha A_z^*\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\right] \exp\left[j\frac{2\pi z}{\lambda}(\gamma-\gamma*)\right] d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right). \tag{13}$$