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A variant on eigenmode method in periodic crossed gratings

Pierre Hillion

Instiut Henri Poincaré, 86 Bis Route de Croissy, 78110 Le Vésinet, France

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Abstract

The eigenvalue problem discussed in Noponen and Turunen [Eigenmode method for electromagnetic synthesis of diffractive elements with three-dimensional profiles, J. Opt. Soc. Am. A 11 (1994) 2494–2502] for crossed gratings is analyzed in a different way using the three components of the electric field instead of the components $E_{x,y}$, $H_{x,y}$. As a result, half as many eigenvectors are needed.

Keywords: Periodic media; Crossed grating; Wave propagation; Modal method

1. Introduction

The methods developed to describe wave propagation in crossed gratings [1] are still confronted to the problem of computer resources [2]. We present a variant of the rigorous theory of binary-surface relief gratings [3] developed to accomodate three-dimensional (3D) modulated profiles and, this variant has the advantage to halve the dimensions of the matrix eigenvalue problem generated by Maxwell's equations in such media. The numerical implantation of the corresponding formalism is not discussed in this theoretical work and we closely follow the notations used in [3], the modulated grazing region periodic in x and y with d_x, d_y periods, respectively, is located in the slab $0<z<h$. We work with the three components of the electric field while the four components $E_{x,y}$, $H_{x,y}$ are used in [2,3].

With the time dependence $\exp(-i\omega t)$ assumed, the Maxwell equations are inside the slab

$$\nabla \wedge E - i\omega\mu_0 H = 0, \quad \nabla \wedge H - i\omega\varepsilon E = 0, \tag{1a}$$

$$\nabla . H = 0, \quad \nabla . \varepsilon E = 0 \tag{1b}$$

and the permittivity ε is [3] with p, q arbitrary integers

$$\varepsilon = \varepsilon_0 \sum_{p,q} \varepsilon_{p,q} \exp[2i(px/d_x + qy/d_y)]. \tag{2}$$

We get from (1a) the Helmholtz equation satisfied by the electric field

$$\Delta E + \omega^2 \mu_0 \varepsilon E - \nabla(\nabla . \varepsilon E) = 0 \tag{3}$$

we look for the solutions of Maxwell's equations in the form, m, n being arbitrary integers

$$\{E, H\}(x) = \sum_{m,n} \{E_{mn}, H_{m,n}\} \phi_{m,n}(x),$$

$$\phi_{m,n}(x) = \exp[i(\alpha_m x + \beta_n y + \gamma z)] \tag{4}$$

with $\alpha_m = 2\pi m/d_x$, $\beta_n = 2\pi n/d_y$. Then

$$\Delta E = -\sum_{m,n} \sum\nolimits_{m,n} \Gamma^2_{m,n} E_{m,n} \varphi_{m,n}(x),$$

$$\Gamma^2_{m,n} = \alpha^2_m + \beta^2_n + \gamma^2, \tag{5a}$$

E-mail address: Pierre.Hillion@wanadoo.fr.

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$$\nabla.\varepsilon E = \mathrm{i}\sum_{m,n}(\alpha_m E_{m,n;x} + \beta_n E_{m,n;y} + \gamma E_{m,n;z})\varphi_{m,n}(x), \tag{5b}$$

$$\begin{aligned}\varepsilon E &= \varepsilon_0 \sum_{m,n}\sum_{p,q}\varepsilon_{p,q}E_{m,n}\varphi_{m,n}(x)\,\exp[2\mathrm{i}(px/d_x + py/d_y)]\\ &= \varepsilon_0 \sum_{m,n}\sum_{p,q}\varepsilon_{m-p,n-q}E_{p,q}\varphi_{m,n}(x). \end{aligned}\tag{5c}$$

Substituting (5a–c) into (3) gives the set of equations in which $k^2 = \omega^2\mu_0\varepsilon_0$

$$\begin{aligned}&\Gamma^2_{mn}E_{mn,x} - k^2\sum_{pq}\varepsilon_{m-p,n-q}E_{pq,x} - \alpha_m(\ldots) = 0,\\ &\Gamma^2_{mn}E_{mn,y} - k^2\sum_{pq}\varepsilon_{m-p,n-q}E_{pq,y} - \beta_n(\ldots) = 0,\\ &\Gamma^2_{mn}E_{mn,z} - k^2\sum_{pq}\varepsilon_{m-p,n-q}E_{pq,z} - \gamma(\ldots) = 0,\end{aligned}\tag{6}$$

$$(\ldots) = \alpha_m E_{pq,x} + \beta_n E_{pq,y} + \gamma E_{pq,z}. \tag{7}$$

Now, let $\Psi_{m,n}$ denote the two-dimensional (2D) vector with the components $\psi_{mn,x}$, $\psi_{mn,y}$

$$\begin{aligned}\psi_{mn,x} &= E_{mn,x} - \alpha_m\gamma^{-1}E_{mn,z},\\ \psi_{mn,y} &= E_{mn,y} - \beta_n\gamma^{-1}E_{mn,z}.\end{aligned}\tag{8}$$

Then substituting into the first two Eqs. (6) the term (…) obtained from the third one and taking into account (8), we get the eigenvalue problem we have now to solve

$$\Gamma^2_{mn}\Psi_{mn} - k^2\sum\nolimits_{pq}\varepsilon_{m-p,n-q}\Psi_{pq} = 0. \tag{9}$$

2. Eigenvalues and eigenvectors

In a numerical implementation of (9), the integers m, p and n, q would be compelled to take finite values from $-M$ to M for m, p; from $-N$ to N for n, q and, changing Σ into Σ' to mark this limitation, we may write (9) ($\delta_{mp}\delta_{nq}$ are Kronecker symbols)

$$\sum_{pq}{}'(\Gamma^2_{pq}\delta_{mp}\delta_{np} - k^2\varepsilon_{m-p,n-q})\Psi_{pq} = 0. \tag{10}$$

Then, we introduce the direct sums

$$r = m\oplus n, \quad s = p\oplus q \tag{11}$$

the integers r, s take $L = (2M+1)(2N+1)$ values and with the notations

$$\begin{aligned}&\Gamma^\dagger_s = \Gamma_{pq}, \quad \delta^\dagger_{rs} = \delta_{mp}\delta_{nq},\\ &\varepsilon^\dagger_{r-s} = \varepsilon_{m-p,n-p}, \quad \Psi^\dagger_s = \Psi_{pq}\end{aligned}\tag{11a}$$

Eq. (10) becomes

$$\sum_{s=1}^{L}A_{r,s}\Psi^\dagger_s = 0, \quad r = 1,2,\ldots,L, \quad A_{r,s} = (\Gamma^\dagger_s)^2\delta^\dagger_{rs} - k^2\varepsilon^\dagger_{r-s} \tag{12}$$

that we may write $A\Psi^\dagger = 0$ with nonnull solutions if $\det A = 0$, a condition supplying L eigen values $(\Gamma_{pq,1})^2$ from which we get according to the definition (5a) of $\Gamma^2{}_{p,q}$, L expressions for γ noted from now on γ_1, $1 = 1,2,\ldots,L$.

As a simple illustration, suppose $M = N = 1$ so that $L = 9$. Since m, n, p, q take the values 1, 0, -1, the nine components of $\Psi^\dagger$ are

$$\Psi^\dagger_1 = \psi_{1,1}, \quad \Psi^\dagger_2 = \psi_{1,0}\cdots\Psi^\dagger_8 = \psi_{-1,0}, \quad \Psi^\dagger_9 = \psi_{-1,-1} \tag{13}$$

while the 9×9 matrix A is

$$\begin{bmatrix}\Gamma^2_{1,1} - k^2\varepsilon_{0,0} & -k^2\varepsilon_{0,1} & -k^2\varepsilon_{0,2} & \ldots & -k^2\varepsilon_{2,1} & -k^2\varepsilon_{2,2}\\ -k^2\varepsilon_{0,-1} & \Gamma^2_{1,0} - k^2\varepsilon_{0,0} & -k^2\varepsilon_{2,0} & \ldots & -k^2\varepsilon_{2,0} & -k^2\varepsilon_{2,1}\\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots\\ -k^2\varepsilon_{-2,2} & -k^2\varepsilon_{-2,1} & -k^2\varepsilon_{-2,0} & \ldots & -k^2\varepsilon_{0,-1} & \Gamma^2_{-1,-1} - -k^2\varepsilon_{0,0}\end{bmatrix}\tag{14}$$

Let Ψ_1 denote the L eigenvectors of (12). Then, with u written for x,y,z we may expand the components $E_{mn,u}$ of the vector field E_{mn}, on the Ψ_1 basis so that

$$E_u(x) = \sum_{mn}{}' E_{mn,u}(x) = \sum_{mn}{}'\sum\nolimits_{l=1}^{L}e_{mnl,u}\phi_{mnl}(x)\Psi_l, \tag{15}$$

$$\phi_{mnl}(x) = \exp[\mathrm{i}(\alpha_m x + \beta_n y + \gamma_1 z)], \tag{15a}$$

with according to (8)

$$\begin{aligned}e_{mnl,x} &= \omega_{mnl} + \alpha_m\gamma_1^{-1}e_{mnl,z},\\ e_{mnl,y} &= \omega_{mnl} + \beta_n\gamma_1^{-1}e_{mnl,z}\end{aligned}\tag{16}$$

so that for m,n fixed, the fields (15) depend on $2L$ arbitrary amplitudes ω_{mnl} and $e_{mnl,z}$.

But these fields are solutions of the Helmholtz Eq. (3) and we have still to impose that they satisfy the divergence Eq. (1b) $\nabla.\varepsilon E = 0$. According to (5c) and (15) we have

$$\sum_{mn}{}'\sum_{pq}{}'\sum_{l=1}^{L}\varepsilon_{m-p,n-q}(\alpha_m e_{pql,x} + \beta_n e_{pql,y} + \gamma_1 e_{pql,z}) \times\phi_{mnl}(x)\Psi_1 = 0 \tag{17}$$

implying

$$\sum_{pq}{}'\varepsilon_{m-p,n-q}(\alpha_m e_{pql,x} + \beta_n e_{pql,y} + \gamma_1 e_{pql,z}) = 0, \quad l = 1,2,3,\ldots L, \tag{17a}$$

which are L constraints on the field amplitudes. Then, for m,n fixed, the electric field has L and on the whole L^2