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A model of evolutionary dynamics with quasiperiodic forcing

Elizabeth Wesson[a,*], Richard Rand[b]

[a]Center for Applied Mathematics, Cornell University, Ithaca 14850, USA
[b]Depts. of Mathematics and Mech. & Aero. Eng., Cornell University, Ithaca 14850, USA

Abstract

Evolutionary dynamics combines game theory and nonlinear dynamics to model competition in biological and social situations. The replicator equation is a standard paradigm in evolutionary dynamics. The growth rate of each strategy is its excess fitness: the deviation of its fitness from the average. The game-theoretic aspect of the model lies in the choice of fitness function, which is determined by a payoff matrix.

Previous work by Ruelas and Rand investigated the Rock-Paper-Scissors replicator dynamics problem with periodic forcing of the payoff coefficients. This work extends the previous to consider the case of quasiperiodic forcing. This model may find applications in biological or social systems where competition is affected by cyclical processes on different scales, such as days/years or weeks/years.

We study the quasiperiodically forced Rock-Paper-Scissors problem using numerical simulation, and Floquet theory and harmonic balance. We investigate the linear stability of the interior equilibrium point; we find that the region of stability in frequency space has fractal boundary.

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Keywords: Replicator equation; quasiperiodic forcing; Floquet theory; harmonic balance; numerical integration.

1. Introduction

The field of evolutionary dynamics combines game theory and nonlinear dynamics to model population shifts due to competition in biological and social situations. One standard paradigm[9,3] uses the replicator equation,

$$\dot{x}_i = x_i(f_i(\mathbf{x}) - \phi), \quad i = 1, \dots, n \tag{1}$$

where x_i is the frequency, or relative abundance, of strategy i; the unit vector $\mathbf{x}$ is the vector of frequencies; $f_i(\mathbf{x})$ is the fitness of strategy i; and ϕ is the average fitness, defined by

$$\phi = \sum_i x_i f_i(\mathbf{x}). \tag{2}$$

* Corresponding author. Tel. +1-601-347-3669.
E-mail address: enw27@cornell.edu

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The game-theoretic component of the replicator model lies in the choice of fitness functions. Define the payoff matrix $\mathbf{A} = (a_{ij})$ where a_{ij} is the expected reward for a strategy i individual vs. a strategy j individual. We assume the population is well-mixed, so that any individual competes against each strategy at a rate proportional to that strategy's frequency in the population. Then the fitness f_i is the total expected payoff for strategy i vs. all strategies:

$$f_i(\mathbf{x}) = (\mathbf{A}\mathbf{x})_i = \sum_j a_{ij} x_j. \tag{3}$$

In this work[10], we generalize the replicator model to systems in which the payoff coefficients are quasiperiodic functions of time. Previous work by Ruelas and Rand[7,8] investigated the Rock-Paper-Scissors replicator dynamics problem with periodic forcing of the payoff coefficients. We also consider a forced Rock-Paper-Scissors system. The quasiperiodically forced replicator model may find applications in biological or social systems where competition is affected by cyclical processes on different scales, such as days/years or weeks/years.

2. The model

2.1. Rock-Paper-Scissors games with quasiperiodic forcing

Rock-Paper-Scissors (RPS) games are a class of three-strategy evolutionary games in which each strategy is neutral vs. itself, and has a positive expected payoff vs. one of the other strategies and a negative expected payoff vs. the remaining strategy. The payoff matrix is thus

$$\mathbf{A} = \begin{pmatrix} 0 & -b_2 & a_1 \\ a_2 & 0 & -b_3 \\ -b_1 & a_3 & 0 \end{pmatrix}. \tag{4}$$

The case $a_1 = \cdots = b_3 = 1$ describes the Rock-Paper-Scissors game as played in real life, and thus can be considered the canonical RPS payoff matrix. We perturb off of this case by taking

$$\mathbf{A} = \begin{pmatrix} 0 & -1 - F(t) & 1 + F(t) \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{5}$$

where the forcing function F is given by

$$F(t) = \epsilon((1 - \delta)\cos\omega_1 t + \delta\cos\omega_2 t). \tag{6}$$

For ease of notation, write $(x_1, x_2, x_3) = (x, y, z)$. The dynamics occur in the 3-simplex S, but since x, y, z are the frequencies of the three strategies, and hence $x + y + z = 1$, we can eliminate z using $z = 1 - x - y$. Therefore, the region of interest is T, the projection of S into the $x - y$ plane:

$$T \equiv \{(x, y) \in \mathbb{R} \,|\, (x, y, 1 - x - y) \in S\}. \tag{7}$$

Thus the replicator equation (1) becomes

$$\dot{x} = -x(x + 2y - 1)(1 + (x - 1)F(t)) \tag{8}$$

$$\dot{y} = y(2x + y - 1 - x(x + 2y - 1)F(t)) \tag{9}$$

Note that $\dot{x} = 0$ when $x = 0$, $\dot{y} = 0$ when $y = 0$, and

$$\dot{x} + \dot{y} = (x + y - 1)(xF(t)(x + 2y - 1) - x + y) \tag{10}$$

so that $\dot{x} + \dot{y} = 0$ when $x + y = 1$, which means that $x + y = 1$ is an invariant manifold. This shows that the boundary of T is invariant, so trajectories cannot escape the region of interest.

It is known[4] that in the unperturbed case ($\epsilon = 0$) there is an equilibrium point at $(x, y) = (\frac{1}{3}, \frac{1}{3})$, and the interior of T is filled with periodic orbits. We see from equations (8)-(9) that this interior equilibrium point persists when $\epsilon \neq 0$. Numerical integration suggests that the Lyapunov stability of motions around the equilibrium point depends sensitively on the values of ω_1 and ω_2. See Figure 1. We investigate the stability of the interior equilibrium using Floquet theory and harmonic balance, as well as by numerical methods.