Available online at www.sciencedirect.com

ScienceDirect

Procedia IUTAM 19 (2016) 161 – 168

Procedia IUTAM

www.elsevier.com/locate/procedia

IUTAM Symposium Analytical Methods in Nonlinear Dynamics

# Solvable cases in the problem of motion of a heavy rotationally symmetric ellipsoid on a perfectly rough plane

Dmitry S. Dobrynin[a], Alexander S. Kuleshov[a,*]

[a]Department of Mechanics and Mathematics, Lomonosov Moscow State University, Leninskie Gory, Moscow, 199991, Russian Federation

**Abstract**

We consider a classical problem of nonholonomic system dynamics – the problem of motion of a rotationally symmetric body on a fixed perfectly rough plane in a case when the moving body is a rotationally symmetric ellipsoid. Using the Kovacic algorithm we found several conditions under which equations of motion of the ellipsoid can be completely solved in quadratures.


Peer-review under responsibility of organizing committee of IUTAM Symposium Analytical Methods in Nonlinear Dynamics

*Keywords:* Rotationally symmetric ellipsoid; Nonholonomic system; Kovacic algorithm; Liouvillian solutions.

## 1. Introduction

The problem of rolling without sliding of a heavy rotationally symmetric body on a fixed horizontal plane is a classical problem of nonholonomic mechanics. In 1897, S.A. Chaplygin in his paper[1] proved that the solution of this problem is reduced to the integration of the second-order linear differential equation with respect to the component of the angular velocity of the body in the projection on its axis of symmetry. However, a solution of this differential equation cannot always be found. In a case when the moving body is a nonhomogeneous dynamically symmetric ball, the general solution of the corresponding equation is expressed in terms of elementary functions[1]. In a case of motion of a circular disk or a hoop on a horizontal plane, the general solution of the mentioned equation is expressed in terms of a hypergeometric series[1]. In the paper[2], Kh.M. Mushtari continued the investigation of the problem of motion of a heavy rotationally symmetric body on a perfectly rough horizontal plane. Under additional condition imposing restrictions on a shape of the rolling body and a mass distribution in it, two new particular cases have been found, when the motion of the body can be investigated completely. In the first case the moving body is bounded by the surface formed by rotating a parabolic arc about an axis passing through its focus, and in the second case the moving body is a rotationally symmetric paraboloid. For other rotationally symmetric bodies moving without sliding on a horizontal plane, an exact solution of the corresponding second-order linear differential equation is unknown.

* Corresponding author. Tel.: +7-495-939-3681 ; fax: +7-495-939-2090.
*E-mail address:* kuleshov@mech.math.msu.su


Peer-review under responsibility of organizing committee of IUTAM Symposium Analytical Methods in Nonlinear Dynamics
doi:10.1016/j.piutam.2016.03.021

In this paper we try to find the exact solution of the corresponding second-order linear differential equation in a case when the moving body is a rotationally symmetric ellipsoid. To find this solution we use the so-called Kovacic algorithm.

In 1986, American mathematician J. Kovacic proposed the algorithm[3] for finding a general solution of a second-order linear differential equation with variable coefficients for a case when this solution can be expressed in terms of so-called liouvillian functions[3,4]. Recall that liouvillian functions are functions that are built up from the rational functions by algebraic operations, taking exponentials and by integration. If a linear differential equation has no liouvillian solutions, the Kovacic algorithm also allows to ascertain that fact.

Using the Kovacic algorithm in the problem of motion of a rotationally symmetric ellipsoid on a perfectly rough horizontal plane we found several cases when the corresponding second-order linear differential equation has liouvillian solutions under additional restrictions on the parameters of the system. Physical admissibility of these additional restrictions is discussed.

The paper is organized as follows. In Section 2 we give the detailed problem formulation following the papers by Chaplygin[1] and Mushtari[2]. We derive also the linear second-order differential equation for a general rotationally symmetric body. In Section 3 we discuss specific features of application of the Kovacic algorithm to second-order linear differential equations. Finally in Section 4 we present our own results obtained in the problem of motion of a heavy rotationally symmetric ellipsoid on a perfectly rough horizontal plane.

## 2. General Problem Formulation

Let us consider the general problem of motion of a rotationally symmetric rigid body on a fixed perfectly rough horizontal plane. Suppose that the centre of mass $G$ of the body is situated on the symmetry axis $G\zeta$, and moments of inertia about principal axes of inertia $G\xi$ and $G\eta$ perpendicular to $G\zeta$ are equal to each other. The body moves in presence of the homogeneous gravity field.

Let $Oxyz$ be the fixed coordinate frame with the origin in the supporting plane $Oxy$. Let $\theta$ be the angle between the symmetry axis of the body and the vertical. The distance $GQ$ of the centre of mass over the plane $Oxy$ is a function of angle $\theta$[1,2]:

$$GQ = f(\theta). \tag{1}$$

Let $\beta$ be the angle between the meridian $M\zeta$ of the body and a certain fixed meridian plane, and $\alpha$ is the angle between horizontal tangent $MQ$ of the meridian $M\zeta$ and the $Ox$-axis. The position of the body is completely determined by the angles $\alpha, \beta$ and $\theta$ and by the $x$ and $y$ coordinates of the point $M$.

Let us specify now the position of the coordinate system $G\xi\eta\zeta$. Suppose that the $G\xi$-axis is always situated in the plane of the vertical meridian $M\zeta$ while the $G\eta$-axis is perpendicular to this plane (Fig. 1). In this case the coordinate system $G\xi\eta\zeta$ moves both in the space and in the body. Denote by $\xi$, $\eta$, $\zeta$ the coordinates of the point of contact $M$ of the body with the supporting plane in the coordinate system $G\xi\eta\zeta$. Then $\eta = 0$ and[1,2]:

$$\xi = -f(\theta)\sin\theta - f'(\theta)\cos\theta, \quad \zeta = -f(\theta)\cos\theta + f'(\theta)\sin\theta, \tag{2}$$

where $()'$ is a derivative of function $f(\theta)$ with respect to $\theta$. Thus we can completely characterize the surface of the moving body using the function $f(\theta)$.

Let the velocity $\mathbf{v}$ of the centre of mass $G$, the angular velocity vector $\boldsymbol{\omega}$ of the body, the angular velocity vector $\boldsymbol{\Omega}$ of the coordinate system $G\xi\eta\zeta$, and the reaction of the plane $\mathbf{R}$ are specified in the system $G\xi\eta\zeta$ by the components $v_\xi, v_\eta, v_\zeta;\ p, q, r;\ \Omega_\xi, \Omega_\eta, \Omega_\zeta$ and $R_\xi, R_\eta, R_\zeta$, respectively. Let $m$ be the mass of the body, $A_1$ – its moment of inertia about axes $G\xi$ and $G\eta$, and $A_3$ – its moment of inertia about the symmetry axis.

Since the $G\zeta$ axis is fixed in the body, then

$$\Omega_\xi = p, \quad \Omega_\eta = q. \tag{3}$$

The third component $\Omega_\zeta$ can be easily expressed through $p$; indeed, the plane $G\xi\zeta$ is always vertical, i.e. the projection of the angular velocity of the axes $G\xi\eta\zeta$ on $MQ$ equals to zero, therefore

$$\Omega_\zeta = \Omega_\xi \cot\theta = p\cot\theta. \tag{4}$$