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Invariants for Parallel Mapping*

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Abstract: This paper analyzes the geometric quantities that remain unchanged during parallel mapping (i.e., mapping from a reference curved surface to a parallel surface with identical normal direction). The second gradient operator, the second class of integral theorems, the Gauss-curvature-based integral theorems, and the core property of parallel mapping are used to derive a series of parallel mapping invariants or geometrically conserved quantities. These include not only local mapping invariants but also global mapping invariants found to exist both in a curved surface and along curves on the curved surface. The parallel mapping invariants are used to identify important transformations between the reference surface and parallel surfaces. These mapping invariants and transformations have potential applications in geometry, physics, biomechanics, and mechanics in which various dynamic processes occur along or between parallel surfaces.

Key words: second gradient operator; second class of integral theorem; parallel mapping; invariants; transformations

Introduction

Recently, a series of mapping invariants called Gauss mapping invariants have been identified for Gauss mapping[1]. Similar invariants also exist for parallel mapping. The purpose of the paper is to derive the invariants for parallel mapping.

Parallel mapping originated from constant thickness shells. For example, thin shell theory is an important branch of solid mechanics which uses parallel surfaces as a classical geometric concept. The middle surface of the thin shell with constant thickness is abstracted as a smooth curved surface A. An arbitrary point p on the curved surface A is depicted by the point vector $\boldsymbol{r}=\boldsymbol{r}(u^1,u^2)$ with u^i being the Gauss parameter coordinates and $\boldsymbol{n}$ the unit normal at point p as shown in Fig. 1. A point p^* along the unit normal with distance $pp^*=z$ is then chosen. Then, the point vector of p^* is

$$\boldsymbol{r}^*(u^1,u^2)=\boldsymbol{r}(u^1,u^2)+z\boldsymbol{n} \tag{1}$$

If point p moves in the surface A with z kept unchanged, the locus of points p^* will form a curved surface A^*. The surface A^* is called the parallel surface of surface A. According to this definition, the

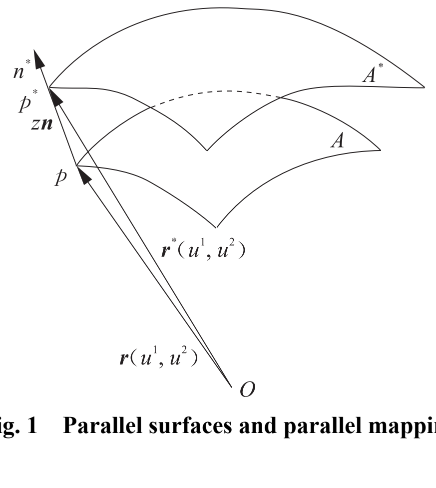

Fig. 1 Parallel surfaces and parallel mapping

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upper and lower surfaces are all parallel surfaces of the middle surface of every shell with constant thickness.

Parallel surfaces exist not only in thin shell structures, but also in many scientific and engineering fields. For example, multi-walled carbon nanotubes in physics and material sciences, cell membranes in cell biology, lipid bilayer vesicles in biophysics, and cables and tubes in modern industries, all include parallel surfaces which may be abstracted and idealized for all these structures. Thus, parallel surfaces exist in many fields, from science to technology, within disciplines and interdisciplinary fields, from macro to micro scales, and from inorganic to organic systems. Therefore, the study of the general characteristics of parallel surfaces is of great importance.

Equation (1) defines a parallel mapping. Once surface A is mapped into surface A^*, the geometric quantities of surface A will also be mapped onto surface A^*. Most of the geometric quantities will be changed during the mapping process and these changes need to be well understood. However, there is another equally important (or even more important) question of which geometric quantities remain unchanged during parallel mapping. Thus, this study will analyze which geometric quantities are parallel mapping invariants.

The starting point for studying parallel mapping invariants can be the recent progress in the mathematics of biomembranes[2] and recent advances in Gauss mapping invariants. The studies of biomembranes lead to the second gradient operator for curved surfaces[3]. The second gradient operator was then used to develop the second class of integral theorems[4-7] and the Gauss-curvature-based integral theorem. The second gradient operator, the second class of integral theorems, the Gauss-curvature-based integral theorem, and Gauss mapping were then combined to indentify a series of Gauss mapping invariants[1]. These ideas are extended here to parallel mapping to get a series of parallel mapping invariants. These invariants are then used to draw geometric transformations between a reference surface and a parallel surface which have various potential applications in various disciplines.

1 Second Gradient Operator and Second Class of Integral Theorems

Before studying the parallel mapping invariants, first recall the fundamental tensors, the second gradient operator, the second class of integral theorems, and the deduced conservative laws.

1.1 Fundamental tensors on the reference surface

On the reference surface A, the fundamental tensors at point p satisfy the tensor equation[8]:

$$\boldsymbol{L}^2 - 2H\boldsymbol{L} + K\boldsymbol{G} = \boldsymbol{0} \tag{2}$$

Here, $\boldsymbol{G}$ is the first fundamental tensor, $\boldsymbol{L}$ is the second, and $\boldsymbol{L}^2 = \boldsymbol{L}\cdot\boldsymbol{L}$ is the third. $H = (c_1 + c_2)/2$ is the mean curvature and $K = c_1 c_2$ is the Gauss curvature with c_1 and c_2 as the two principle curvatures. Equation (2) is similar to an algebraic equation with rank two and also has two solutions, $\boldsymbol{L}$ and $\hat{\boldsymbol{L}}$. They also satisfy the Viete theorems:

$$\hat{\boldsymbol{L}} + \boldsymbol{L} = 2H\boldsymbol{G} \tag{3a}$$

$$\hat{\boldsymbol{L}}\cdot\boldsymbol{L} = K\boldsymbol{G} \tag{3b}$$

where $\hat{\boldsymbol{L}}$ is the conjugate fundamental tensor of $\boldsymbol{L}$.

1.2 Second gradient operator and second class of integral theorems

The fundamental tensors $\boldsymbol{G}$ and $\hat{\boldsymbol{L}}$ may be used to define differential operators:

$$\nabla(\cdots) = \boldsymbol{g}_i g^{ij} \frac{\partial(\cdots)}{\partial u^j}, \quad i, j = 1, 2 \tag{4a}$$

$$\bar{\nabla}(\cdots) = \boldsymbol{g}_i \hat{L}^{ij} \frac{\partial(\cdots)}{\partial u^j}, \quad i, j = 1, 2 \tag{4b}$$

Here, $\boldsymbol{g}_i$ is the covariant base vector. g^{ij} and $\hat{L}^{ij}$ are the contravariant components of $\boldsymbol{G}$ and $\hat{\boldsymbol{L}}$. ∇ is the first gradient operator in conventional differential geometry[8] and $\bar{\nabla}$ is the second gradient operator[3-7]. In differential geometry, the first gradient operator ∇ leads to the first class of integral theorems[5,6]. In the same way, the second gradient operator $\bar{\nabla}$ leads to the second class of integral theorems for a tensor field $\boldsymbol{T}$[6,7]:

$$\iint_A \mathrm{d}A\bar{\nabla}\otimes\boldsymbol{T} = \oint_C \mathrm{d}\boldsymbol{s}\cdot\hat{\boldsymbol{L}}\otimes\boldsymbol{T} - \iint_A 2K\mathrm{d}\boldsymbol{A}\otimes\boldsymbol{T} \tag{5a}$$

$$\iint_A \mathrm{d}\boldsymbol{A}\times\bar{\nabla}\otimes\boldsymbol{T} = \oint_C \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{L}\otimes\boldsymbol{T} \tag{5b}$$

where "$\otimes$" is a unified operator. If "$\otimes$" in Eq. (5a) is eliminated, the result is the second gradient theorem. If "$\otimes$" is replaced by inner product "$\cdot$", the result is the second divergence theorem. If "$\otimes$" is replaced by exterior product "$\times$", the result is the second curl theorem for the tensor field. Equation (5b) may be regarded