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Steady-State Performance of Kalman Filter for DPLL

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Abstract: For certain system models, the structure of the Kalman filter is equivalent to a second-order variable gain digital phase-locked loop (DPLL). To apply the knowledge of DPLLs to the design of Kalman filters, this paper studies the steady-state performance of Kalman filters for these system models. The results show that the steady-state Kalman gain has the same form as the DPLL gain. An approximate simple form for the steady-state Kalman gain is used to derive an expression for the equivalent loop bandwidth of the Kalman filter as a function of the process and observation noise variances. These results can be used to analyze the steady-state performance of a Kalman filter with DPLL theory or to design a Kalman filter model with the same steady-state performance as a given DPLL.

Key words: Kalman filter; digital phase-locked loop (DPLL); steady-state performance

Introduction

Digital phase-locked loops (DPLLs) are widely used for carrier or spread-code tracking, symbol synchronization, or timing recovery in various communication systems. Since these problems are essentially phase estimation problems, Kalman filter theory can also be used in these applications[1-6]. For some system models, the Kalman filter has the same structure as second-order variable gain DPLLs[1,2]. The use of Kalman filters instead of DPLLs makes it possible to simultaneously achieve fast acquisition time, wide acquisition range, and low tracking jitter, without the tradeoffs associated with conventional fixed-gain DPLLs[1,2,6].

For precise system models, the Kalman filter is a linear minimum mean square error (LMMSE) estimation, which gives both the estimate and the estimate error. However, in practice, the precise system model is impossible to determine. Thus, the Kalman filter performance is difficult to evaluate. In contrast, DPLLs are widely used and the appropriate loop bandwidth can be easily chosen for various schemes. Since the structures of Kalman filters and second-order DPLLs are the same, the steady-state Kalman gain should be equivalent to the second-order DPLL gain. To apply the knowledge of DPLLs to the design of Kalman filters, the relationship between the Kalman filter model and DPLL loop bandwidth should be studied. Patapoutian gave an exact solution for the steady-state Kalman gains with respect to the noise variances and initial parameters[5]. However, the form is too complex to illustrate the relationships. Therefore, a simple approximate solution is needed to clarify the relationships.

This paper shows that the steady-state Kalman gain has the same form as the DPLL gain. This paper also presents an approximate simple solution of the steady-state Kalman gain with an expression for the equivalent loop bandwidth of the Kalman filter as a function of the process and observation noise variances. These results can be used to analyze the steady-state performance of Kalman filters with DPLL theory. The result can also be used in reverse to determine the noise variances in the Kalman filter model for a given equivalent loop bandwidth to achieve rapid acquisition without loss of tracking reliability.

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1 Kalman Filter Model for DPLL

Denote ϕ_n and f_n as the phase and Doppler shift of the signal to track at sample *n*, respectively, and *T* as the sample period. Define the state vector as $\boldsymbol{s}_n = (\phi_n \quad Tf_n)^{\mathrm{T}}$ and x_n as the observed signal phase. The dynamic system can then be modeled as

$$\boldsymbol{s}_n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \boldsymbol{s}_{n-1} + \begin{pmatrix} 0 \\ u_n \end{pmatrix} = \boldsymbol{A}\boldsymbol{s}_{n-1} + \boldsymbol{U}_n \tag{1}$$

$$x_n = (1 \quad 0)\boldsymbol{s}_n + \omega_n = \boldsymbol{h}\boldsymbol{s}_n + \omega_n \tag{2}$$

where $u_n \sim N(0, \sigma_Q^2)$ and $\omega_n \sim N(0, \sigma_n^2)$ represent the process and observation noise[1].

Define $\boldsymbol{G}_n = (G_{0,n} \quad G_{1,n})^{\mathrm{T}}$ as the Kalman gain, $\boldsymbol{K}$ as the error variance matrix, $\hat{\boldsymbol{s}}_n = (\hat{\phi}_n \quad T\hat{f}_n)^{\mathrm{T}}$ as the *n*-th estimate, and $\hat{\boldsymbol{s}}_{n|n-1}$ as the *n*-th prediction from sample *n*−1. Note that $\varepsilon_n = x_n - \boldsymbol{h}\hat{\boldsymbol{s}}_{n|n-1}$. The Kalman equations with the initial conditions $\hat{\boldsymbol{s}}_{-1}$ and $\boldsymbol{K}_{-1|-1}$ [7] are as follows:

$$\boldsymbol{K}_{n|n-1} = \boldsymbol{A}\boldsymbol{K}_{n-1|n-1}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q} \tag{3}$$

$$\boldsymbol{G}_n = \frac{\boldsymbol{K}_{n|n-1}\boldsymbol{h}^{\mathrm{T}}}{\boldsymbol{h}\boldsymbol{K}_{n|n-1}\boldsymbol{h}^{\mathrm{T}} + \sigma_n^2} \tag{4}$$

$$\boldsymbol{K}_{n|n} = \boldsymbol{K}_{n|n-1} - \boldsymbol{G}_n\boldsymbol{h}\boldsymbol{K}_{n|n-1} \tag{5}$$

$$\hat{\boldsymbol{s}}_{n|n-1} = \boldsymbol{A}\hat{\boldsymbol{s}}_{n-1} \tag{6}$$

$$\hat{\boldsymbol{s}}_n = \hat{\boldsymbol{s}}_{n|n-1} + \boldsymbol{G}_n\varepsilon_n \tag{7}$$

where $\boldsymbol{Q} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_Q^2 \end{pmatrix}$.

Equation (7) can then give the following expression for the signal phase prediction:

$$\hat{\phi}_{n+1|n} = \hat{\phi}_{n|n-1} + T\hat{f}_{n|n-1} + G_{0,n}\varepsilon_n + G_{1,n}\varepsilon_n \tag{8}$$

The block diagram representation of Eq. (8) shown in Fig. 1 is equivalent to a second-order DPLL with a proportional integral, except for the time-varying Kalman gains instead of the fixed gains in the DPLL[1,2].

2 Steady-State Equivalent Loop Bandwidth of Kalman Filter

This section proves the equivalence between the converged Kalman gain and the DPLL gain and then develops a relationship between the noise variances for the Kalman filter and the DPLL loop bandwidth.

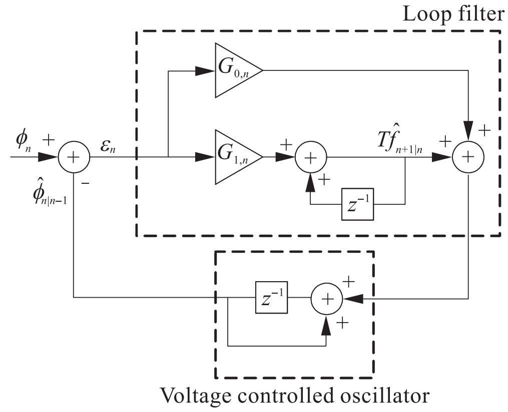

Fig. 1 Kalman filter block diagram

When $n \to \infty$, the Kalman filter converges to the the steady state, which means that $\boldsymbol{K}_{n+1|n} = \boldsymbol{K}_{n|n-1} = \boldsymbol{K} = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$, $\boldsymbol{G}_{n+1} = \boldsymbol{G}_n = \boldsymbol{G}$. These equations can be substituted into Eqs. (3)-(5) to get the followings (as shown in the Appendix):

$$K_{00}^4 = \sigma_Q^2(K_{00} + \sigma_n^2)(K_{00} + 2\sigma_n^2)^2 \tag{9}$$

$$\boldsymbol{G} = \begin{pmatrix} \dfrac{K_{00}}{K_{00} + \sigma_n^2} & \dfrac{\sigma_Q}{\sqrt{K_{00} + \sigma_n^2}} \end{pmatrix}^{\mathrm{T}} \tag{10}$$

Defining $\omega = \dfrac{\sqrt{2}K_{00}}{T(K_{00} + 2\sigma_n^2)}$ as the equivalent natural frequency, the steady-state Kalman gain has the form,

$$\boldsymbol{G} = \begin{pmatrix} \dfrac{8\dfrac{1}{\sqrt{2}}\omega T}{4 + 4\dfrac{1}{\sqrt{2}}\omega T} & \dfrac{4\omega^2T^2}{4 + 4\dfrac{1}{\sqrt{2}}\omega T} \end{pmatrix}^{\mathrm{T}} \tag{11}$$

According to Ref. [8], the gain of second-order DPLL is

$$(C_1 \quad C_2) = \frac{1}{k_0k_1}\begin{pmatrix} \dfrac{8\zeta\omega T}{4+4\zeta\omega T+(\omega T)^2} & \dfrac{4(\omega T)^2}{4+4\zeta\omega T+(\omega T)^2} \end{pmatrix} \tag{12}$$

where ζ denotes the damping ratio, and k_0 and k_1 denote the gains of the discriminator and the numerical controlled oscillator (NCO). Comparison of Eqs. (11) and (12) shows that the steady state Kalman gain is equivalent to the DPLL gain with $\zeta = 1/\sqrt{2}$ and $k_0k_1 = 1$, except for the $(\omega T)^2$ term in the denominator. In the DPLL, the sampling time and the natural frequency should fulfill $\omega T \ll 1$ to function properly[8,9]. Thus, $(\omega T)^2$ is a very small quantity, which