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Analytical Expression for the MIMO Channel Capacity*

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Abstract: This paper presents analytical expressions for the multiple-input multiple-output (MIMO) channel capacity in frequency-flat Rayleigh fading environments. An exact analytical expression is given for the ergodic capacity for single-input multiple-output (SIMO) channels. The analysis shows that the SIMO channel capacity can be approximated by a Gaussian random variable and that the MIMO channel capacity can be approximated as the sum of multiple SIMO capacities. The SIMO channel results are used to derive approximate closed-form expressions for the MIMO channel ergodic capacity and the complementary cumulative distribution function (CCDF) of the MIMO channel capacity (outage capacity). Simulations show that these theoretical results are good approximations for MIMO systems with an arbitrary number of transmit or receive antennas. Moreover, these analytical expressions are relatively simple which makes them very useful for practical computations.

Key words: multiple-input multiple-output (MIMO) system; ergodic capacity; outage capacity; analytical expression; Rayleigh fading channel

Introduction

Information theory has demonstrated the enormous potential capacity of wireless communication systems with antenna arrays at both the transmitter and the receiver[1,2], when the channel exhibits rich scattering and complete channel state information (CCSI) is available at the receiver.

Although expressions have been derived for the multiple-input multiple-output (MIMO) channel capacity for various conditions, most cases are for the ideal situation where the CCSI is available at the receiver[3,4] and there are no simple analytical expressions for practical computations. Some analytical expressions have been derived for the MIMO channel ergodic capacity and outage capacity, but the expressions are very complex which limits their practical use. For instance, the ergodic capacity was obtained for Rayleigh MIMO channels using the integral of the eigenvalues of the channel matrix[2,5-7], but numerical integrations are still required to calculate the results so they are too complex for practical use. It has been shown that a limiting value of capacity scaling exists when both the transmit and receive antennas tend to infinity[8], but this result is of little use for a limited number of antennas. For Rayleigh fading channels, the complementary cumulative distribution function (CCDF) of the capacity was studied using the Monte Carlo simulation[1,9,10]. Molisch et al.[11] extracted the parameters of the multipath components and used a synthetic variation of their phases to evaluate the MIMO channel outage capacity from measured data, but the measurements are cumbersome. A Gaussian approximation to the MIMO channel capacity was used to obtain a good approximation to the CCDF of Rayleigh MIMO channel

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capacity[5,6]. However, the first two moments of the Gaussian variable were still obtained from numerical integration of the eigenvalues of the channel matrix; therefore, these results are too complex for practical use. Zhang et al.[12,13] presented an approximate expression for the ergodic capacity of Rayleigh fading single-input multiple-output (SIMO) channels, and their expression of the ergodic capacity for Rayleigh fading MIMO channels is still more complex and less accurate than the present results. Furthermore, the outage capacity is not concerned in these papers.

Simple and accurate analytical expressions are needed not only for the ergodic capacity but also for the outage capacity for MIMO channels, since they are very useful for evaluating the channel capacity. This paper first presents an exact analytical expression for the SIMO channel ergodic capacity. The MIMO channel capacity is then approximated as the sum of multiple SIMO capacities to derive a simple analytical expression, which is a very good approximation of the actual ergodic capacity. The MIMO channel capacity can also be approximated as a Gaussian random variable to derive an analytical expression for the CCDF of the MIMO channel capacity. Simulation results show that these expressions are good approximations for MIMO systems with an arbitrary number of transmit or receive antennas.

1 System Model

Consider a communication link over a wireless MIMO channel with N transmit and M receive antennas in a frequency-flat Rayleigh fading environment. The transmitter has no channel information; thus, the power is allocated equally to each transmit antenna. CCSI is available at the receiver. Such a system can be described as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v} \tag{1}$$

where $\boldsymbol{x} \in \mathbf{C}^{N\times 1}$ is the transmitted complex signal vector, the entries x_i $(i = 1,\cdots,N)$ are independent and identically distributed (i.i.d.) complex Gaussian variables and $x_i \sim \mathcal{CN}(0, P_{\mathrm{T}}/N)$, where P_{T} is the total transmission power. $\boldsymbol{H} \in \mathbf{C}^{M\times N}$ is the channel matrix and the propagation gains h_{ij} $(i = 1,\cdots,M$; $j = 1,\cdots,N)$ are i.i.d. complex Gaussian variables, $h_{ij} \sim \mathcal{CN}(0, 1/N)$. The additive white Gaussian noise $\boldsymbol{v} \in \mathbf{C}^{M\times 1}$ has i.i.d. entries $v_i \sim \mathcal{CN}(0, \sigma_n^2)$ (i= $1,\cdots,M$). $\boldsymbol{y} \in \mathbf{C}^{M\times 1}$ is the received signal vector.

2 Analytical Expression for SIMO Channel Ergodic Capacity

When CCSI is available at the receiver, i.e., there is no channel estimation error, the MIMO system ergodic capacity is given by[1, 2]

$$C_{\text{Erg-MIMO}}(N, M, \rho) = \mathrm{E}_{\boldsymbol{H}}\left\{\log_2\left|\boldsymbol{I}_M + \frac{P_{\mathrm{T}}}{N\sigma_n^2}\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}\right|\right\} = \mathrm{E}_{\boldsymbol{H}}\left\{\log_2\left|\boldsymbol{I}_M + \rho\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}\right|\right\} \tag{2}$$

where $\boldsymbol{A}^{\mathrm{H}}$ denotes the Hermitian conjugate transpose matrix of $\boldsymbol{A}$, $\mathrm{E}\{\bullet\}$ denotes expectation and $\rho = P_{\mathrm{T}}/(N\sigma_n^2)$ is the average SNR at each receive antenna.

N=1 describes a SIMO system. The channel matrix can be expressed as a complex Gaussian vector $\boldsymbol{h} \in \mathbf{C}^{M\times 1}$ with $\boldsymbol{h} \sim \mathcal{CN}(\mathbf{0}_{M\times 1}, \boldsymbol{I}_M)$. From Eq. (2), the ergodic capacity of a SIMO system with M receive antennas is

$$C_{\text{Erg-SIMO}}(M, \rho) = \mathrm{E}_{\boldsymbol{h}}\left\{\log_2\left|\boldsymbol{I}_M + \frac{P_{\mathrm{T}}}{\sigma_n^2}\boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}\right|\right\} = \mathrm{E}_{\boldsymbol{h}}\left\{\log_2\left|\boldsymbol{I}_M + \rho\boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}\right|\right\} \tag{3}$$

The analytical expression for the SIMO channel capacity from Eq. (3) is given by the following theorem.

Theorem 1 When CCSI is available at the receiver, the SIMO system ergodic capacity can be expressed as a function of the number of receive antennas M and the average SNR ρ at each receive antenna. Its analytical expression is

$$C_{\text{Erg-SIMO}}(M, \rho) = \frac{1}{\ln 2}\left(1 + \sum_{i=1}^{M-1}\frac{1}{(M-i)!}\left(-\frac{1}{\rho}\right)^{M-i}\right) \times \mathrm{e}^{\frac{1}{\rho}}E_1\left(\frac{1}{\rho}\right) + \frac{1}{\ln 2}\sum_{i=1}^{M-1}\sum_{k=1}^{M-i}\sum_{l=1}^{k}(-1)^{M-i-k} \times \rho^{i+l-M}\frac{1}{k\times(M-i-k)!\times(k-l)!} \tag{4}$$

where $k!$ is the factorial of a nonnegative integer k. $E_1(x)$ is the exponential integral, $E_1(x) = \int_1^{\infty}\mathrm{e}^{-xt}t^{-1}\mathrm{d}t$.

Equation (4) is the exact closed-form capacity